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"You use your imagination:" An investigation into how students use 'imaging' during numeracy activities

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Abstract

Developing knowledge about how students acquire mathematical understanding is a focus of mathematics curricula and research, including the ability of students to move from manipulating concrete materials to abstract number properties when solving problems. This study, informed by the Numeracy Development Projects (Ministry of Education 2007a, 2008b) and the work of Pirie and Kieren (1989, 1992, 1994a, 1994b), examines the role of 'imaging' in supporting the development of students' mathematical thinking and understanding. Imaging is an important phase of the teaching model advocated by the Numeracy Development Project. The context of this study is a primary school mathematics programme, which involved the teaching of two mathematics units that focused on the addition and subtraction of decimal fractions and whole numbers. There is considerable research about what is effective in mathematics education for diverse learners, and how students learn. There is, however, very limited research about the role of imaging in mathematical learning.

This qualitative study adopted a case study approach and focused on a group of Year 6 students. Data collection methods included observation, interviews, field notes and document analysis. A thematic approach was used to analyse data and to develop and inform an emerging theoretical framework.

During this study I developed a model, entitled A Model for the Development of Students' Mathematical Understanding, which illustrates six mathematical resources students use as they solve problems. These resources are: *materials*, *mental picture images*, *drawn picture images*, *transformed mental images*, *transformed drawn images* and *number properties*. Students' engagement with these six resources illustrates how they develop understanding of mathematical concepts. The students identified a preference for using drawn rather than mental images when solving problems. This study also emphasizes the complexity of the imaging process, and the fluid and multifaceted nature of learning in mathematics. This study serves to highlight the complexities of the teaching and learning process in mathematics for both teachers and students.

Glossary

Basic facts: the sums and products of all pairs of whole numbers from 0-9.

Change unknown: an equation where the second number is unknown.

Compatible numbers: numbers that add together to make a tidy number (a multiple of ten), often ten.

Compensation: when one number is rounded to a tidy number (a multiple of ten), and an adjustment is made to another number, for example 28 + 36 as 30 + 34.

Decimat: a rectangular piece of paper that can be divided (by folding or ruling) to represent decimal fractions, usually tenths, hundredths and thousandths. It is designed to support the teaching of decimal place value.

Decipipes: materials designed to teach decimal place value, and decimal addition and subtraction strategies. They consist of 'whole' pipes, and segments representing tenths, hundredths and thousandths. The thousandths are represented by metal rings.

Empty number line: a line with no scale or beginning or end numbers, but which respects the order of numbers.

Derived facts: students use a set of known basic facts to find or derive answers to an unknown problem.

Hundreds board: materials designed to support the development of patterning and part-whole concepts. The reverse side of the board includes only some of the numbers to encourage imaging.

Knowledge domain: information a student should be able to recall without needing to work it out.

Linked cubes: cubes that can be joined or separated to represent numbers or equations.

Place value decomposition: separating a number according to the value of the places in a number.

Shielding: part of the Using Imaging phase of the Strategy Teaching Model. Materials are shielded (usually behind a screen) and students describe how they would manipulate them if able.

Strategy: a way of working out the answer to a problem or part of a problem.

Tidy number: multiples of ten (ending in a zero).

Venn diagram: overlapping circles that illustrate simple set relationships.

Chapter 1: Introduction

This study focuses on the role of imaging in solving mathematical problems and in the development of mathematical knowledge. The context for the study is the Strategy Teaching Model advocated by the Numeracy Development Project, and used in New Zealand primary schools.

The focus on numeracy in New Zealand mathematics curricula since the 1990s can be seen against the background of similar curriculum developments and reform in a number of countries, including the United Kingdom and Australia (Ministry of Education, 1992, 2007a; Walls, 2004). In all three countries, concerns about declining standards in traditional subjects, including arithmetic, and poor student performance in international tests were coupled with beliefs about the need to ensure citizens were capable of functioning in the modern world, and about the importance of mathematics as a "key engine in the economy" (Anthony & Walshaw, 2007, p. 5; Walls, 2004). These concerns led to the introduction of numeracy programmes that aimed to raise student performance (Higgins, 2003b). In New Zealand, the Numeracy Development Projects (NDP) have focused on teaching and learning in number and algebra as central to the pathway to numerical literacy (Ministry of Education, 2008b).

Over the last twenty years mathematics curricula advocates have focused on teaching mathematics with understanding, and on students making sense of mathematical ideas (Pirie & Kieren, 1994b). In New Zealand, this has been translated into a view that the mathematics taught and learned should "provide a foundation for working, thinking, and acting like mathematicians and statisticians" (Anthony & Walshaw, 2007, p. 6). Mathematics teaching and learning has traditionally involved the use of concrete materials and abstract numbers. It is the ability of students to make the move from manipulating concrete materials to solving mathematical problems using abstract concepts that has been a long-term concern in international literature (Bobis, 1996; Moyer, 2001; Shuhua, Kulm, & Wu, 2004). To function mathematically, it is assumed that students must be able to solve problems using abstractions (number properties). Twenty years ago von Glasersfeld commented that using concrete materials is useful in moving towards these abstractions. However, he cautioned, the materials, "no matter how ingenious they might be, merely offer an opportunity for actions from which the desired operative concepts may be abstracted." Further, it cannot be assumed that students will move automatically from manipulating materials to solving problems using abstractions (von Glasersfeld, 1992). A similar observation was made by Moyer, writing in 2001. He stated that, "It is a false assumption to believe that mathematical relationships are abstracted from empirical objects" (Moyer, 2001, p. 192). Like von Glasersfeld, he suggested that it is necessary to connect students' own internal representations with the external representations of the equipment.

The Numeracy Development Projects (NDP) have an aim of fostering students making the connection between manipulating materials and solving problems using abstract number properties, and with understanding. The NDP have a number of curriculum aspects including a mathematical content framework based on eight stages of learning, a teaching model for developing strategies, and a range of teaching and student resources for classroom use. The inclusion within the teaching model of the concept of imaging is a deliberate attempt to establish a connecting "bridge" between concrete materials and abstractions (Ministry of Education, 2008b, p. 7). Imaging has a role both as a 'stage' within the Framework (Stage 3), and as one of the three phases of the Strategy Teaching Model (Ministry of Education, 2007b, 2008b). Students working at Stage 3, Counting from one by imaging, solve addition and subtraction problems by imaging "visual patterns of objects in their mind and [counting] them" (Ministry of Education, 2007b, p. 3). Imaging is also one of the three phases of the Strategy Teaching Model (STM): Using Materials, Using Imaging and Using Number Properties. Progress through these phases "demonstrates greater degrees of abstraction in a student's thinking," as existing knowledge is turned into new strategies and knowledge (Ministry of Education, 2008b, p. 5). The phases of the STM are related to the Pirie and Kieren theory (Pirie & Kieren, 1989). Both the Strategy Teaching Model and the Pirie and Kieren theory will be discussed in more detail in Chapter 2.

The teaching and learning model outlined in the STM raises a number of questions about how students learn mathematics. How do they make sense of the problems and strategies they are presented with and asked to use? How do they approach and solve problems? What does imaging mean to students? What are the thinking processes students use as they work towards Using Number Properties? Finally, does the STM reflect the ways students actually solve problems?

One of the main catalysts for my interest in imaging is that I have noted that many teachers are less clear about the purposes and aims of imaging than about other aspects of the NDP. As a classroom teacher, I was initially unclear about the purposes of imaging. In my role as the lead teacher of numeracy in my school and as a numeracy adviser, I have had many conversations with teachers who expressed similar concerns to the ones I felt, some even commenting that they left the imaging phase out. These concerns seem to be counter to material issuing from the developers of the NDP that "teachers seem to readily accept the *Using Images* phase as being an obvious and natural stage that has frequently been missing in their teaching" (Hughes, 2002, p. 356). Imaging is one of the main features of the NDP, and therefore central to questions about teaching and learning in mathematics. The following sections discuss the background to the development of the NDP and their place within the mathematics curriculum.

1.1 Background to the Numeracy Development Projects

The Numeracy Development Projects (NDP) were developed and implemented as a result of recommendations from the 1997 Mathematics and Science Taskforce (Higgins, 2003b), which was established in response to the 1995 Third International Mathematics and Science Study (TIMSS). The TIMSS results showed that New Zealand students were achieving significantly below international averages, with students performing poorly in number (including place value, fractions and computation), measurement and algebra concepts (Higgins, 2003b). Another factor leading to the Taskforce's establishment was feedback from the mathematics teaching community about difficulties implementing the 1992 curriculum, particularly the problem-solving approach required by the curriculum (Higgins, 2003b). In addition, many teachers still appeared to be teaching in a transmission style, dictated by the textbooks they used, rather than using the constructivist principles that underpinned the philosophy of the mathematics curriculum (Holmes & Tozer, 2004). The Mathematics and Science Taskforce highlighted a number of priorities in relation to improving mathematics performance, including the need to develop the pedagogical knowledge of teachers, improve quality teaching and teacher confidence, and provide resources and professional development to support mathematics teaching and learning (Higgins, 2003b).

Introduced as professional development programmes for teachers, the NPD have the objective of improving student learning and achievement through the development of teacher capability (Anthony & Walshaw, 2007). The projects were influenced by international numeracy projects, particularly the Count Me in Too (CMIT) project in New South Wales (Ministry of Education, 2008b), but also Cognitively Guided Instruction research and place value studies (Higgins, 2003a; Holmes & Tozer, 2004). Adapted from the Mathematics Recovery Programme developed by Wright, the goal of CMIT was "providing teachers with better understanding of young children's mathematical thinking and ways of developing more sophisticated mathematical thinking in their students" (Wright, 2000, p. 146). CMIT included a research-based framework for the teaching and assessment of number concepts in the early years of school, which was developed as a result of small-scale, intensive observations of students' words and actions as they solved problems (Wright, 1998, 2000). This framework became the basis for the early stages of the strategy section of the NDP Number Framework (Ministry of Education, 2008b). However, CMIT was restricted to the first three years of school, so it was necessary to extend it for use with older students in New Zealand, initially students up to the end of Year 8, but later extended to Year 9 and 10 students (Ministry of Education, 2008b). The NDP were introduced to teachers through professional development workshops and in-class visits facilitated by numeracy advisers. These introduced key numeracy resources, including a diagnostic assessment tool to access students' thinking, the Strategy Teaching Model, the Number Framework, teaching resource materials, and concrete materials to use when making problems during the Using Materials phase. Classroom organization and management was also a focus of these workshops, particularly the ability grouping of students for instruction, the introduction of a rotation for teaching groups, the structure of lessons, and the role of discourse. The professional development workshops and in-class support also aimed to give teachers knowledge about how students acquire number concepts and increase their understanding of how to assist students' progress. The realities of the NDP, when enacted in school contexts, were sometimes different. Students in a class could not be fitted neatly into the three ability groups it was said should be the maximum within each class, timetable pressures meant it was not possible to teach numeracy five times a week, and teachers found it difficult to find the time needed to plan and prepare for numeracy lessons. One teacher I worked with while a numeracy adviser said that all he wanted to know about numeracy was how to plan his week's programme in less than 30 minutes. While his 30-minute timeframe may have been unrealistic, it does illustrate the tensions and frustrations felt by many teachers.

1.2 The NDP as the enacted mathematics curriculum

Over the past 15 years, since the introduction of the NDP, the term numeracy has become almost synonymous with mathematics in New Zealand primary schools (Begg, 2006; Walls, 2004). Begg (2006) comments that the introduction of the NDP was based on two assumptions: that numeracy is the basic building block of mathematics, and that numeracy includes more than just number. While Begg (2006) acknowledges that numeracy is important, he says it should not be the dominant aspect of the primary school mathematics curriculum. Nevertheless, the 2007 curriculum, the introduction of National Standards, and the revision to the National Administration Guidelines (giving priority to teaching numeracy in Years 1-8) seem to have strengthened the place of numeracy as the de facto mathematics curriculum (Ministry of Education, 2007a, 2009). The Venn diagrams in the 2007 curriculum document clearly illustrate the emphasis to be placed on the number and algebra strand at all levels of primary schooling (the same Venn diagrams are used in the National Standards document), further reinforcing the importance of number and algebra (Ministry of Education, 2007a, 2009; see Appendix A).

In turn, it could be argued that the NDP have become the de facto number and algebra curriculum. Although the NDP were introduced as teacher professional development programmes, they have become much more; with their resources, planning and assessment materials, the NDP have become the cornerstone of many teachers' mathematics programmes. Young-Loveridge (2009) and Scouller (2009) both comment on teachers' reliance on NDP resource materials, and suggest this may be due to lack of teacher confidence. However, this reliance could also be a result of the way the NDP are presented – stages matched to curriculum levels, a complete, ready-to-teach programme that, if followed, is assumed to ensure that most of the *New Zealand Curriculum's* number and algebra achievement objectives are taught (Ministry of Education, 2007a).

The focus on mathematics education and professional learning for teachers has continued in recent years. In addition to changes to the content and structure of the mathematics curriculum (renamed

Mathematics and Statistics), the 2007 *New Zealand Curriculum* (*NZC*) saw the introduction of the concept of 'Teaching as inquiry,' with "teachers required to inquire into the impact of their teaching on their students" (Ministry of Education, 2007a, p. 35). The importance of teachers' pedagogical knowledge is also a key theme in the *Best evidence synthesis* (Anthony & Walshaw, 2007), a document that analyses international research literature to deepen understanding about what is effective in mathematics education for diverse learners, and how students learn. The importance of discourse is discussed, for example, ensuring students have the opportunity to work with and learn from each other, to articulate their thinking, and respectfully share ideas. The role of teachers in questioning, clarifying, and fine-tuning mathematical thinking is considered. The importance of mathematical tasks is also highlighted, including the use of realistic contexts, open-ended tasks, linking tasks to students' prior knowledge, and the use of 'tools' as learning supports. Thus, there has been a considerable focus on both inquiry into how students learn in mathematics, and the pedagogy that supports effective teaching and learning. However, the role of imaging in mathematical learning does not appear to have been discussed. In my view, this suggests that imaging is a taken for granted but largely unexamined mathematical process.

1.3 Researcher background

I have been a primary teacher for fourteen years, teaching students from Years 0 to 8 in three Christchurch schools. In 2006 when I participated in professional development in the Advanced Numeracy Project, and since then have used the NDP as key resources in my mathematics programmes. That year, I became a lead teacher for numeracy, supporting teachers in my school to implement the NDP in their classrooms, and to develop school-wide resources and curriculum programmes.

In 2007 I conducted a small-scale research project into imaging. I carried out an investigation involving four students in my Year 5 class over a period of three weeks, teaching learning outcomes from Stage 6 involving proportions, ratios and fractions. I gathered data during teacher-led group activities and discussions with students (both individually and in the group), and I observed the interactions between students as they worked together to solve problems. This study was an opportunity to trial data collection methods, including the use of semi-structured interviews and field notes. My findings suggested that students image as part of the modelled process taught during teacher-led activities, and when problems become too difficult to solve with just abstract numbers. The study left me with a number of questions. Would the findings vary if the group were larger? If similar research were undertaken with groups of varying mathematical ability, would the results vary? And would the result vary with a different age group?

In the intervening years, I have changed schools and have taught Year 3 and 5 students, been a team leader of both year groups, and a lead teacher for mathematics. I was also seconded for a term as a numeracy adviser with University of Canterbury Education Plus. In this role I worked with teachers in Years 1-6 in Canterbury and Nelson, and took part in professional development, including a regional hui for all numeracy advisers in the South Island. This has provided me with further insights into both the NDP and the role of imaging, and raised additional dilemmas about teachers' confidence teaching the imaging stage, and how far the use of mental images supports the acquisition of number properties (Pirie, 2002).

I wanted to explore further the question of imaging, particularly how (and if) students use imaging as they solve problems. I also wanted to find out more about the images students create, and when they use imaging. Finally, I wanted to try to find out more about the role imaging plays in helping students develop the knowledge to be able to solve increasingly difficult problems, and problems using number properties. My research questions reflect my aims:

When do students image as they solve mathematical problems?

What images do students create when solving mathematical problems?

How do students use imaging when solving mathematical problems?

How does imaging influence students' abilities to solve increasingly difficult problems?

1.4 Outline of the thesis

Chapter 2 presents an overview of the literature related to teaching and learning of mathematics in New Zealand. I discuss and critique the Numeracy Development Projects, such as teaching numeracy in the classroom, the Strategy Teaching Model and imaging. I also discuss the aims of the study. *Chapter 3* sets out the research methodology for the study, followed by a description of the research design, including ethical considerations, the data collection methods used, changes to and limitations of the research design, and the data analysis process. *Chapters 4 and 5* report the analysis of the data, which is organized around five themes. *Chapter 6* discusses the results and includes links to the literature. It is in this chapter that I focus on the term ‘mathematical resources’ to encompass students’ activities, including imaging, materials and number properties, as they solve problems. *Chapter 7* concludes the thesis. Here I present the conclusions I have reached as a result of the research, put forward a model to illustrate the development of mathematical understanding, before concluding with the implications of the study.

Chapter 2: Literature review

2.1 Introduction

This chapter provides an overview of the relevant literature and comprises literature that discusses the background to teaching and learning mathematics in New Zealand, and the Numeracy Development Project, including the Number Framework and diagnostic and assessment materials. Literature relevant to the teaching of numeracy in the classroom is discussed, including the grouping of students, the role of discourse, the use of authentic contexts to present problems, and mental calculation and written recording. Literature relevant to the Strategy Teaching Model of the Numeracy Development Projects is also discussed, including literature discussing the role of imaging. Finally, this chapter describes the aims of this study.

2.2 Teaching and learning in mathematics

Teaching and learning mathematics in New Zealand is based on a constructivist model. The 1992 curriculum was influenced by two international reports (the Cockcroft Report from the UK, and the National Council of Teachers of Mathematics Standards from the USA), which articulated a constructivist approach to mathematical learning (Ell, 2001). The curriculum introduced context-bound problem solving and a focus on students' mathematical thinking (Ell, 2001). In addition, the curriculum document acknowledged the importance of connecting new mathematical concepts and skills to prior knowledge and skills, and building on this prior experience (particularly for Māori and female students) (Ministry of Education, 1992, p. 12). The constructivist principles of the 1992 curriculum have continued into the Numeracy Developments Projects (NDP) and the 2007 *New Zealand Curriculum* (Ministry of Education, 2007a, 2008b). Features of constructivist theories will be discussed, together with important research related to two key studies, Cognitively Guided Instruction and place value. These are based on constructivist principles, and have influenced the development of the NDP (Ministry of Education, 2008b).

2.2.1 Constructivist theories of teaching and learning

Constructivism places the individual at the centre of an active, constructive process of learning (Cobb, 2007; von Glasersfeld, 1990). Learners construct knowledge, which, following the principles advocated by Piaget, is built on previous learning (Bobis, Mulligan, & Lowrie, 2004; von Glasersfeld, 2000). No individual is a 'blank slate;' all individuals have ways of dealing with their experiential environments. The prior knowledge a learner has influences the development of new knowledge, and is the only basis on which new knowledge can be built (Confrey, 1990; von Glasersfeld, 1993). Furthermore, "learning is an active constructive process" (Wood, 2002, p. 61). Knowledge cannot be transferred to the learner; rather it has to be built up by the learner, who actively selects and rejects information from previous experiences. In doing this, the learner continually reorganizes knowledge (von Glasersfeld, 1990, 1993).

Within constructivism there are a number of contrasting perspectives, including radical and social constructivism. Inherent in radical constructivism is the notion that it is impossible to check whether our ideas and concepts correspond to external reality. Further, reality is unknowable because it exists in human thought and action (Cobb, 2007). Therefore, no knowledge is unique – a solution to a problem can never be regarded as the only possible solution (von Glasersfeld, 1993). A constructivist teacher can "never justify what she teaches by claiming it is 'true.' In mathematics, she can show that the logic derives from certain conventional operations" (von Glasersfeld, 1993, p.

34). Even a concept such as 3 + 3 can only be considered certain because convention has constructed it in that particular way.

In contrast, social constructivism sees learning as linked to the social setting in which it takes place (Cobb, 2007). Learning is thus a social process, negotiated in collaboration with other students and the teachers who are part of classroom learning communities (Bobis et al., 2004; Brophy, 2006). Wood (2002) observes that rich social interactions with others contribute to students' opportunities for learning. For these interactions to occur, there must be an environment where communication is central (Bobis et al., 2004; Hunter, 2006). In this way individual students actively contribute to the evolution of classroom mathematical practices, for example, how groups of tens and ones are counted either using materials or mentally. Cobb (2000) describes the relationship between students' individual activity and classroom practices as "one of reflexivity" (p. 173). Classroom practice and individual student activities are interrelated, one is not derived from the other: "individual students are seen to contribute to the evolution of classroom mathematical practice as they reorganize their mathematical understanding" (Cobb, 2000, p. 173). In this way students' mathematical behaviour is enabled by their participation in mathematical practices. The ways students reorganize their understandings, however, is also constrained by the norms and practices developed in the classroom (Cobb, 2000). Thus interpretations that fit with those understood by society generally are developed (Cobb, Wood, & Yackel, 1990). The implication of this, as both Bobis et al. (2004) and Hunter (2006) comment, is that there needs to be discussion, and sharing of ideas, explanations and opinions, and that this is usually facilitated and scaffolded by the teacher.

Studies by Nuthall (Brophy, 2006; Nuthall, 1999, 2004) suggest a number of limitations of social constructivism. One assumption of social constructivism is that all students are active participants in classroom discussions. However, Nuthall noted that it is more common for a group of students to be highly active and another group to be mostly silent. In his classroom research, Nuthall (1999) found that the nature of the talk students engage in can be problematic. He established that about 25 per cent of what students learn about academic concepts and principles comes from input they receive from peers rather than the teacher, and that some of this may be vague, distorted or incorrect (Nuthall, 2004). Nuthall's findings led him to draw two major conclusions about the successful implementation of social constructivist teaching. The first is that such teaching works best when the focus of discussion is about experiences students have shared with each other because the language and activity are mutually supportive, thus enabling shared meaning. The second is that the aims of social constructivist teaching are much easier to achieve in small groups, where all are likely to participate fully in the group process, and where it is more likely that there are shared meanings (Brophy, 2006).

Constructivism, as a theory of learning, has a number of implications for teaching. Students are not 'blank slates,' but have prior learning and knowledge, which is the only basis on which they can learn (von Glasersfeld, 1993). Therefore teachers need to understand students, identify their prior learning, assist students to restructure their ideas, and relate new learning to that already in students' repertoires (Hunter, 2006; von Glasersfeld, 1993). It is not simply enough to show the student the 'correct' answer (Confrey, 1990). Hughes, Desforges and Mitchell (2000) caution that understanding students' intellectual development, and arranging experiences that challenge and extend it, is not easy, and may be resisted by students. During the last twenty years a number of studies have attempted to provide teachers with connected experiences to challenge and extend students. One of these studies is described below.

2.2.2 Cognitively Guided Instruction

Cognitively Guided Instruction (CGI) is an approach to teaching and learning school mathematics which combines a teacher development programme based on constructivist ideas of understanding students' thinking in a problem-solving context, and using this understanding to develop more advanced mathematical ideas, with a research-based model of the development of students' mathematical thinking (Fennema et al., 1996). CGI is not an instructional programme; it does not provide teachers with materials or explicit guidelines about how they teach students (Fennema et al., 1996).

The CGI programme includes both workshops and in-class support for teachers. It focuses on helping teachers to expand and organize their understanding of students' thinking, and to use this to plan appropriate activities. The programme provides teachers with a framework for interpreting students' strategies that enables them to develop their knowledge of mathematics and the curriculum (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996). By developing a detailed knowledge about students' thinking, teachers are able to refocus their thinking about pedagogy, "so that the primary considerations revolve around student thinking rather than teacher actions" (Carpenter et al., 1996, p. 16).

The emphasis in CGI is on how students' informal or intuitive ideas form the basis of more formal concepts and procedures (Fennema et al., 1996). Teachers plan teaching to build on this informal understanding as they guide students to use more effective strategies and more complex mathematical representations through problem-solving and reflective discussions about mathematical situations (Fennema et al., 1996).

The CGI framework incorporates addition, subtraction, multiplication and division problems within a structure of strategy stages that sees students moving from the direct modelling of problems to using number properties (Anthony & Bicknell, 2002). A key theme in the CGI framework is that "children intuitively solve word problems by modeling the action and relations described in them" (Carpenter et al., 1996, p. 6). Over time these direct-modelling strategies, using various physical materials, are replaced by counting strategies, which are actually abstractions of the direct-modelling strategies (Carpenter et al., 1996; Fennema et al., 1996). As their knowledge develops further students use invented procedures, using addition, subtraction or derived facts, and the "words they use to describe their manipulations of blocks become the solutions themselves" (Carpenter et al., 1996, p. 13). The emphasis moves from counting to operations using abstractions. The modelling and counting solutions students used with smaller numbers are extended to multi-digit problems. Students use physical materials, which they then abstract, before inventing their own multi-digit algorithms (Carpenter et al., 1996; Fennema et al., 1996). Students' understanding of place value is learned as "they invent procedures to solve problems that require regrouping and counting by 10s" (Fennema, Carpenter, & Franke, 1992, p. 5).

Cognitively Guided Instruction is one of the studies that informed the development of the Numeracy Development Projects (Holmes & Tozer, 2004). Holmes and Tozer (2004) comment that, among other influences, the "Numeracy Project has its roots in…principles of cognitively guided instruction" (p. 61). For example, the CGI stages of strategies have parallels in the NDP. Both programmes emphasize use of word problems, the importance of mathematical thinking and the discussion of alternative strategies (Anthony & Bicknell, 2002). The structure and content of the teacher professional development programmes also have similarities, with workshops and in-class support for teachers, and an emphasis on changing teaching practices (Anthony & Bicknell, 2002; Carpenter et al., 1996; Fennema et al., 1996).

CGI provides teachers with a framework to understand the development of students' whole-number concepts, as they move from using physical objects to counting to derived-fact strategies. It also describes the development of place value understanding. As students develop increasingly efficient ways to solve problems, their understanding of place value increases; students acquire the skills and understanding required to solve problems at the same time as they solve the problems (Carpenter et al., 1996). The development of place value understanding is further discussed in the following section.

2.2.3 Place value

The concept of place value is a key principle informing the development of the Number Framework of the NDP (Ministry of Education, 2008b). *Book 3: Getting started* notes that progression between Early Part-Whole (Stage 5) and Advanced Additive Part-Whole (Stage 6) presents a significant challenge to students. This is because "students' understanding of place value…requires them to perform operations on numbers rather than merely identify digits in columns" (Ministry of Education, 2008b, p. 9). In order to emphasize the link between place value and operations, Book 5 is entitled *Teaching addition, subtraction, and place value*, and place value thinking is included in most activities (Ministry of Education, 2007c, 2008b).

One of the reasons for the emphasis of place value understanding was that the 1997 TIMSS study showed this was one of the areas of weakness for New

Zealand students. The development of the Framework was informed by work by Fuson et al. (1997) and Young-Loveridge (1999a, 1999b), both of whom developed models to explain the acquisition of number concepts (Higgins, 2003a; Ministry of Education, 2008b). Fuson found that American students tended to interpret multi-digit numbers as single-digit numbers adjacent to each other rather than recognizing their place value (1992; Fuson et al., 1997). She identified the need for students to understand and construct multi-unit conceptual structures that enabled them to understand the English word name (especially the 'teen' and 'ty' numbers) and the positional base-ten written digits and the relationship between the two (Fuson, 1992). Fuson et al. (1997) found that students who participated in classes where learning about multi-digit concepts was treated as problem-solving activities, showed significant gains in understanding, particularly when they were encouraged to come up with their own strategies.

Young-Loveridge's (1999a, 1999b) study into the development of place value understanding drew on Fuson's work. Young-Loveridge (1999a) highlights the importance of place value, commenting that "Children need to understand place value if they are to use the number system accurately and efficiently to solve problems involving large numbers" (p. 4). She developed a framework for teachers to understand students' thinking and learning about number, beginning with a unitary concept of numbers and extending to include multi-digit numbers. Young-Loveridge emphasized the importance of students having experience with a wide variety of materials (particularly those that can be grouped and ungrouped) to help them appreciate the additive composition of numbers, the significance of the number ten, and the way large numbers are composed of multiples of ten (1999a). She also highlighted the importance of having a variety of activities and representations to explore the same concept, and of students talking about their mathematics (Young-Loveridge, 1999b).

She concluded that the latter may be a key factor in ensuring that students develop an understanding of place value (Young-Loveridge, 1999b).

2.3 A focus on numeracy in New Zealand

The concern around teaching and learning in mathematics in New Zealand in the 1990s which led to the development of the Numeracy Development Projects, and influence of Count Me In Too have been briefly outlined in Chapter 1. This section discusses and critiques aspects of the NDP in more detail.

The NDP sit within the mathematics and statistics learning area of the *New Zealand Curriculum* (Ministry of Education, 2007a). This learning area includes the number and algebra, geometry and measurement, and statistics strands. Within the mathematics and statistics learning area, number and algebra is the dominant strand (see Appendix A for a breakdown of the time spent on each strand at different curriculum levels). It is interesting that in the description of the number and algebra strand, there is no mention of numeracy. The link between the *NZC* and the NDP, however, can be found, implicitly, in the alignment between the achievement objectives for curriculum levels and those in the NDP. The emphasis on numeracy is reinforced by the National Administration Guidelines, which state that priority should be given to numeracy (and literacy) teaching in Years 1 to 8 (Ministry of Education, n.d.).

A number of authors have questioned the emphasis on numeracy in New Zealand (Begg, 2006; Walls, 2004). Walls (2004) observes that the terms mathematics and numeracy are used almost synonymously. She compares the NDP with British and Australian models of numeracy, which incorporate mathematical knowledge and skills from measurement and statistics in addition to number and algebra. She comments that, "Numeracy in these countries is seen as a form of broad mathematical literacy" (2004, p. 28). In contrast, the NDP "present a much more limited vision of numeracy as constituting skills and knowledge exclusively related to number and numerical calculation" (Walls, 2004, p. 25). Writing two years later, Begg (2006) questioned the emphasis on numeracy, arguing that while number is important in mathematics, much more than number is needed for mathematical success. This emphasis on numeracy, however, is not apparent in the *Best evidence synthesis*, a key document about mathematical research. In fact, it is striking to notice how few references there are to the Numeracy Development Projects (Anthony & Walshaw, 2007).

In my experience, the terms numeracy and mathematics have been used almost synonymously in recent years by the New Zealand mathematics education communities. In many classrooms, including those in my school, numeracy is taught rather than number and algebra. Many schools refer to their mathematics curriculum leaders as 'lead teachers of numeracy,' as do professional development and research findings (see, for example, Ministry of Education, 2010). It also seems that numeracy has been largely taught in isolation to other strands of mathematics and other curriculum areas, despite comments in the *New Zealand Curriculum* about linking learning areas (Ministry of Education, 2007a). There are a number of possible explanations for this, including the organization of the NDP resource materials into discrete areas covering the three strategy domains, and the limited resources linking numeracy with other mathematics strands. The following section discusses the Numeracy Development Projects in more detail because of their importance in terms of teaching programmes in New Zealand schools.

2.4 The Numeracy Development Projects

The objective of the NDP is to improve student learning and achievement through the development of teacher capability (Anthony & Walshaw, 2007). Professional development programmes, facilitated by specialist numeracy advisers, were designed to provide teachers with knowledge about how students acquire number concepts, and increase their understanding of how to assist students' progress. To facilitate this, a diagnostic tool was developed by the writers of the project to provide an effective means of accessing students' thinking (Hunter, 2006). Thomas, Tagg and Ward (2002) reported that most teachers involved in the NDP say their knowledge has developed though involvement with the project.

Central to the NDP is the Number Framework. The following section provides an explanation and discussion of the Framework as a model for teaching and learning.

2.4.1 The Number Framework

The Number Framework is the link between the NDP and the *NZC*; it "embodies most of the achievement aims and objectives" in Levels 1 to 5 of the number and algebra learning area of the *NZC* (Ministry of Education, 2007b, p. 1). The Framework is divided into two sections: strategy and knowledge. The strategy section "describes the mental processes students use to estimate answers and solve operational problems with numbers," and the knowledge section, "the key items of knowledge that students need to learn" (Ministry of Education, 2007b, p. 1). Students must make progress in both sections of the framework: "strong knowledge is essential for students to broaden their strategies…and is often an essential prerequisite for the development of more advanced strategies…[and] more advanced strategies help students to develop knowledge" (Ministry of Education, 2007b, p. 1). The strategy section of the Framework was developed on the basis of research (including Wright, 1998) that investigated students' arithmetical thinking. This research found identifiable common progressions in the development of number concepts. The strategy section consists of a sequence of global stages, presented as an inverted triangle to represent that expansion in knowledge and range of strategies students have available as they progress through the stages (Johnston, Thomas, & Ward, 2010; Ministry of Education, 2007b; Appendix B). Each stage is distinct and sets out increasingly sophisticated thinking (Johnston et al., 2010). Students (and teachers) move from one strategy to the next, following the NDP planning materials (available on the nzmaths website). The sequential learning represented by the stages is criticized by Walls (2004) as reinforcing the idea that learning is linear, predictable and beneficial because students move towards "progressively 'smarter' stages" (p. 37). She argues that the "rigidly structured linear learning progressions… creates or reinforces beliefs and expectations about children's being 'at' a certain stage by a certain age" (Walls, 2004, p. 37). This belief has been reinforced by the introduction of National Standards (Ministry of Education, 2009). However, the Framework emphasizes that, although students build new strategies on existing strategies, the existing strategies are not subsumed. Rather students are able to revert to previous strategies when faced with unfamiliar problems or when the mental load is too high (Ministry of Education, 2007b).

Walls has also criticized the separation of the Framework into distinct strategy and knowledge sections. She states that the division appears to "present a view of mathematical knowledge and strategies as separate" (Wall, 2004, p. 36). Earlier, the main writer, Hughes (2002) acknowledged that critics who argued that the division was arbitrary may have a point. What starts as strategy for a student will become knowledge, making the division between the two artificial. However, he justified the maintenance of the dichotomy for pedagogical reasons: very different teaching models are required for teaching the two aspects of the framework (Hughes, 2002). This is because students learn knowledge so they can recall it automatically, but reason with numbers to solve strategy problems (Ministry of Education, 2007b, 2008b).

Since the inception of the NDP in 2001, there have been other responses based on both empirical studies and critique. These are discussed in the following section.

2.4.2 Literature related to the NDP implementation

Young-Loveridge (2009) analysed data gathered from students whose teachers had participated in the first year of the NDP in 2003, 2005 or 2007 (almost 250,000 students). She found that, when gains made in the first year of professional development were analysed in terms of effect size, the NDP programme "has produced substantial gains in terms of progress on the Framework" (Young-Loveridge, 2009, p. 28). In fact, "most of the gains from NDP professional development were above average, and many of these gains were excellent" (Young-Loveridge, 2009, p. 28).

The data also show that, despite these gains, students are still not reaching the levels thought to be necessary if they are to reach an acceptable level of achievement by Year 12 (Young-Loveridge, 2009). This data is interesting, because, although a great deal has been written about the NDP (including the annual research and evaluation reports published by the Ministry of Education), there appear to be very few articles that raise questions about the structure and implementation of the NDP. Young-Loveridge and other commentators suggest a number of reasons for these findings. One possibility is that the two-year professional development programme may not be long enough to enable all teachers to "acquire the deep and connected understanding" they need to ensure more students reach the expected achievement levels (Young-Loveridge, 2009, p. 30). It appears that many teachers are still too reliant on the NDP resource books. Many teachers may still lack confidence using the NDP materials (Young-Loveridge, 2009). Scouller (2009) also comments on teachers' reliance on NDP materials, and suggests that their use as 'text books' by some teachers may reflect a lack of understanding. However, in my experience, teachers' reliance on the NDP materials may also result from the emphasis placed on using the teaching resource books during professional development workshops, and in the planning materials on the nzmaths website. Teachers are given an unspoken message that they are expected to rely on the teaching and planning resources.

There are other concerns about the NDP. First, the structure of the Framework (with its four counting and four 'advanced' stages) implicitly emphasizes the importance of counting. Teachers may spend too long teaching the first four stages, or be reluctant to move students on to Stage 5 even though they are beginning to demonstrate that they are using part-whole thinking. This is despite the statement that teachers' "major objective is to assist students to understand and use part-whole thinking as soon as possible…. [It] is important that you [teachers] realize that this is the aim for all number teaching at all levels of schooling" (Ministry of Education, 2008b, p. 9). This may be one reason for students using the counting stages for longer than desirable. Another reason may be that students are reluctant to move away from the proven reliability of counting (Young-Loveridge, 2009). Second, the structure of the programme, together with the reliance on NDP teaching resources, may mean that students are proficient at solving problems within the confines of the NDP examples, but do not develop a broader understanding of the concepts which would enable them to solve problems presented within different contexts or settings (Bloomfield, 2003).

Finally, the structure of the Framework, with its alignment between knowledge and strategy, may foster notions of 'learning a stage.' The Framework structure may suggest to teachers that students must have all the required knowledge at a stage before moving to the next stage for strategy teaching, rather than continuing to develop knowledge, while working on the next strategy stage (Young-Loveridge, 2009). Scouller (2009) also critiques the Framework, in particular the balance between knowledge and strategy. She suggests that, in many classes, strategy teaching overrides knowledge development, and that strategy is seen as the goal of learning rather than a tool to develop understanding. Scouller (2009) argues that strategies are useless if students have not acquired the appropriate bank of knowledge. I have seen this recently in my classroom, with students understanding how to find fractions of sets using multiplication and division, but unable to solve the problems because of a lack of basic facts knowledge.

2.5 The NDP in practice

"The Numeracy Project suggests a language based model based on teachers listening and questioning students' justifications and explanations" (Woodward & Irwin, 2005, p. 799). This has implications for the way teachers organize and implement their numeracy programmes. This section discusses those aspects of the organization for teaching that are relevant to this study: the diagnostic and assessment tools, the nature of grouping for numeracy teaching, the role of discourse, the use of authentic contexts when framing problems, and the roles of mental calculation and written recording.

2.5.1 Diagnostic and assessment tools

The NDP assessment tools are "designed to give quality information about the knowledge and mental strategies of the students you work with" (Ministry of Education, 2008a, p. 1). The principle assessment tool, the NumPA test, consists of a series of interview questions that are aligned to the Number Framework. It is designed to help the teacher understand the strategic thinking that is going on in their students' heads, in order to be able to determine the most appropriate strategy stages for the students in their class (and to highlight knowledge gaps) (Ministry of Education, 2008b). The format of interviews with individual students is used to help teachers understand where students are in terms of their knowledge (von Glasersfeld, 1993). This is because, "teachers, who have the goal of changing something in students' heads, must have some notion of what goes on in those other heads" (von Glasersfeld, 1992, p. 3).

I used interviews with individual students in both the initial interviews and final assessments. However, I did not use the NumPA assessment because this would have only provided a snapshot of their thinking when solving a range of problems. I wanted detailed information on how they solved problems involving the specific strategies that had been taught during the units. For this reason, I used a modified version of a school-wide assessment task developed to provide both formative and summative data about the specific strategies related to each stage of the Framework.

2.5.2 Grouping

The principles of social constructivism are evident in the emphasis on small group teaching within the NDP. Higgins (2006) comments that this emphasis is to enable students to individually and collectively manipulate materials and give all students the opportunity to participate in discussions. Students also have opportunities to articulate their thinking and understanding, and to exchange and critically test ideas, without everyone in the class listening to and watching what is being said (Walshaw & Anthony, 2006).

Both the organization of the teaching resources in Books 5, 6 and 7 (according to the global stages of the Framework) and the guidance provided in *Book 3: Getting started* suggest that, for most of the time, teaching will take place in ability groups (Ministry of Education, 2007c, 2007d, 2008b, 2008c). *Book 3: Getting started* states that teachers' "initial grouping of students should be by their dominant strategy stage" (Ministry of Education, 2008b, p. 10). This makes it "easier for you [the teacher] to pose problems that are broadly in the student's 'zone of proximal development'" (Ministry of Education, 2008b, p. 11). In this way teachers are able to develop programmes that "tightly match the next progression in their [the students'] learning trajectory" (Ministry of Education, 2008b, p. 12). However, *Book 3: Getting started* does caution that the "exclusive use of ability groups can limit students' expectations of themselves" (Ministry of Education, 2008b, p. 12).

The use of ability grouping in mathematics has been debated by a number of authors. They comment on the detrimental effects ability grouping may have on the development of students' mathematical dispositions and sense of mathematical identity (Anthony & Walshaw, 2007; Boaler, 2009; Walls, 2004). Boaler cites a UK government review, which found that "The adoption of structured ability groupings has no positive effects on attainment but has detrimental affects on the social and personal outcomes for some children" (Blatchford, Hallam, Ireson, Kutnick, & Creech, 2008, as cited in Boaler, 2009, p. 97). Instead of ability groups, Boaler (2009) advocates the use of mixed ability groups where the teacher 'opens' the work to make it suitable for all levels, and which enables students to work to the highest level they can reach.

The result of such grouping is that work is at the right level and right pace for all students. However, the organization of the NDP teaching resources, with learning outcomes and activities matched to specific stages, would make this difficult for most teachers.

2.5.3 The role of discourse

The nature of classroom discourse is addressed both explicitly and implicitly in the NDP (Woodward & Irwin, 2005). Emphasis is put on students discussing, explaining and justifying their thinking, with each other as well as with the teacher, often using think-pair-share. After private thought, students share ideas with their neighbour and then with the group. This strategy lends credibility to thinking, fosters mathematical communicating, and helps students develop confidence (Reis & Garvin, 1999). The teacher facilitates discourse by questioning and guiding students in order to elicit their mathematical thinking. In addition, the teacher models mathematical language both in their questioning and through revoicing (expanding, rephrasing or clarifying) an answer (Woodward & Irwin, 2005).

The benefits of encouraging mathematical explanation are discussed by a number of authors. Hunter (2009) observes that the active participation required of all students represents a move (for many) away from being passive receivers of information towards engaging in collaborative interaction and productive discourse. The most common rationale is that students engage in reasoning and thinking as they solve problems, rather than just giving the answer. In this way they become familiar with the conventions of mathematical discussion, including inference, analysis and modelling, and enhance their views of themselves as mathematical learners and doers (Walshaw & Anthony, 2008).

The teacher's role is seen as pivotal in both establishing the norms of discourse, and facilitating and mediating interactions (Anthony & Walshaw, 2007; Fraivillig, Murphy, & Fuson, 1999; Hunter, 2009). The teacher guides discussions by eliciting, supporting and extending student responses (using strategies including questioning and revoicing), and in this way clarifies, encourages and extends mathematical thinking (Fraivillig et al., 1999; Hunter, 2009; Walshaw & Anthony, 2008; Woodward & Irwin, 2005). When students are supported to examine a range of strategies, and to justify their thinking and probe their mathematical ideas, they are initiated into mathematical conventions, which they can then use as tools for developing and communicating their own thinking (Hunter, 2009; Lampert, 1990). The teacher's pedagogical content knowledge and pedagogical expertise is also important for the quality of teacher-student interaction. Effective teachers are able to pick up critical moments within a discussion and take the learning forward by drawing out key mathematical ideas. They make connections between language and conceptual understanding, and introduce students to the conventions that regulate mathematical practice, such as inference, modelling and analysis (Anthony & Walshaw, 2007; Fraivillig et al., 1999). Making connections with students' experiences, and building on prior knowledge, is also important in the choice of the contexts used to present problems. This will be discussed in the following section.

2.5.4 The use of realistic contexts to present mathematical problems

Problems are presented using a variety of realistic contexts in the NDP teaching materials. Although contexts do not necessarily have to be 'real,' they do need to facilitate students thinking in 'real,' purposeful ways (Watson, 2004). To achieve this, problem contexts need to be within students' range of experience and accessible to all. They should also interest and motivate students. It is also important that contexts illustrate a range of possible applications. Finally, the problem, and the context within which it is presented, is assumed to provide opportunities for mathematical reasoning and thinking, and support student understanding (Anthony & Walshaw, 2003, 2007; Irwin, 2001).

The use of contexts, however, does not automatically lead to improved outcomes for all students (Sullivan, Zevenbergen, & Mousley, 2002). Forbes (2000, as cited in Anthony & Walshaw, 2003, p. 53) found that Maori students appeared to be disadvantaged by the contexts into which problems were put because the contexts did not necessarily make sense. Contexts can also obscure the purpose of the task for some students by focusing on the contextual rather than mathematical aspects of problems (Lubienski, 2000). The language demands of contextual problems can also be a barrier to some students (Anthony & Walshaw, 2007). This has implications for teachers using the NDP. To ensure that students understand the purpose of the task, teachers may need to spend time helping students to decode problems, identify key mathematical phrases and translate these into equations. One way of doing this is to encourage students to create contexts for problems, enabling them both to put the problem into a context that makes sense to them, and use appropriate language. In discussing ways in which the problems presented within authentic contexts can be solved, the NDP emphasizes the importance of mental calculation and oral reasoning. These are described below, together with the role of written recording.

2.5.5 Two elements of solving problems: mental calculation and written recording

The NDP puts considerable emphasis on "flexibility and facility with mental calculation" in solving problems, commenting that "algebraic thinking is assisted by flexible mental calculation" (Ministry of Education, 2008b, p. 3). The description and discussion of material in the teaching resources, and the use of small group discussion, emphasize mental calculation and oral reasoning, mediated by the teacher and the group. Mental calculation is also a key element of the diagnostic interview and other assessment resources.

The role of written recording (which is categorized as one of the knowledge strands) is less clear, as evidenced by two apparently contradictory statements. In *Book 1: The Number Framework*, written recording is described as a

> thinking tool, a communication tool, and a reflective tool. It is critical that, at every stage, students engage in building meaning through recording their mathematical ideas in the form of pictures, diagrams, words, and symbols. Making explicit links between oral and written forms is fundamental to the structure of the language of mathematics. (Ministry of Education, 2007b, p. 14)

In contrast, *Book 3: Getting started* outlines that "access to written recording systems can help students to store information, but mental working space is the critical constraint" as they solve problems (Ministry of Education, 2008b, p. 10). A further contradiction appears to exist between the descriptions of written recording and the strategy stage known as Using Imaging. Although written recording is described as a thinking tool in *Book 1: The Number Framework*, it does not appear to be promoted in the description of the Using Imaging phase in *Book 3: Getting started.* Here students are encouraged to orally describe how they manipulated imagined picture images of materials to get the correct answer (Hughes, 2002; Ministry of Education, 2008b).

Mental calculation has been a focus of many curricula for the last twenty-five years. The Realistic Mathematics Education programme in the Netherlands also sees mental calculation as the foundation for the further development of flexible computation and problem-solving strategies (Klein, Beishuizen, & Treffers, 1998). Mental calculation not only stimulates conceptual understanding and procedural proficiency, but also number sense and the understanding of number relations. However, in solving problems, "calculation could be done not only 'in the head' but also by 'using one's head' in that the use of written work was encouraged" (Klein et al., 1998, p. 444). Written recording is not transformed into written calculation; rather by writing, students display the flexible thought processes that are essential to mental calculation.

One model for supporting calculation 'using one's head' is the empty number line. Introduced in The Netherlands with the aim of developing greater flexibility in mental arithmetic, it is a blank number line with no numbers or markers (van den Heuvel-Panhuizen, 2008). This allows students to explore addition and subtraction problems, beginning where they want to on the line and moving up and down in jumps (Ell, Smith, Stensness, & Major, 2010). In this way the empty number line is a model for representing mathematical solutions, which both facilitates solution procedures and allows students to communicate their procedures (Klein et al., 1998). The linear nature of the empty number line supports the development of informal strategies, because, unlike a structured number line, it is not equated with a rigid ruler that has fixed, pre-given distances (Klein et al., 1998).

The empty number line is a constructivist tool, enabling students to build on what they know. Using it gives students opportunities to develop their own strategies, and to develop more sophisticated and abstract strategies, through 'progressive mathematization' (during which students' ideas become more like formal mathematics) (Beishuizen, 2010; Ell et al., 2010; Klein et al., 1998). A key feature of the empty number line is its ability to enhance the flexibility of mental thinking, and to enable students to move to more efficient and flexible ways of solving problems (Beishuizen, 2010). Another characteristic is that students are cognitively involved in their actions; they are solving the computation task within the problem at the same time as recording their jumps (Klein et al., 1998).

The empty number line acts as a scaffold: "Marking the steps on the number line functions as a kind of scaffolding: it shows what part of the operation has been carried out and what remains to be done" (Klein et al., p. 447). In this way it helps students to keep track of what they are doing, leading to a reduction of memory load (Klein et al., 1998). Written work on the empty number line also has a secondary function, "supporting or recording the strategies chosen as mental decisions in the first place," in a flexible way (Beishuizen, 2010, p. 179). The success of the empty number line depends on how it is used (van den Heuvel-Panhuizen, 2008). If used rigidly or wrongly in a prescribed way (for example, step-by-step calculations with no flexibility for shortcuts), its use can be detrimental to students' understanding and proficiency in mental calculation.

While the empty number line appears to link students' understanding with written symbolism, recordings of intuitive understandings do not always lead to formal written symbolism (Young-Loveridge, Taylor, Hāwera, & Sharma, 2007). In a study of the strategies a group of Year 7-8 students used to solve an addition problem involving unlike fractions, these authors found that, of the small number of students who drew diagrams (all circular representations), most were unable to use these to solve the problem correctly. Warner (2003, as cited in Anthony & Walshaw, 2007, p. 127) also highlights tensions between students' informal ways of symbolizing and notation, and formal notations, commenting that in some cases students' attempts to adopt formal schemes acted as a barrier to the development of their mathematical understanding.

In New Zealand, the role of written recording is seen to support "the development of mathematical concepts by encouraging reflective abstraction" (Ell et al., 2010, p. 212). Students use their own ways of recording ideas to build meaning, and gradually move from informal to formal methods of recording. In this way they are given a scaffold to help them progress towards the NDP strategy stage of Using Number Properties (Ell et al., 2010).

Written recording does not always have to be written by individual students. Modelling books are a way of creating a shared written record of thinking (Ell et al., 2010; Higgins, 2006). The NDP promotes the use of the modelling book because it fulfils a number of purposes: it is a reference point and a focal point for discussions, and may refresh memories about previous discussions (Higgins, 2005). Students can also refer back to previous problems and solutions, and this can deepen thinking and discussion. All group participants contribute their thinking to the written record, which helps to enrich everyone's understanding. It is suggested, that, as students see their ideas valued, they are likely to be encouraged to share other ideas and strategies (Ell et al., 2010; Higgins, 2006).

The use of written recording in the modelling book can also provide a link between physical models and mathematical abstractions, by making connections between the materials and the numbers and symbols recorded to represent the way the materials have been manipulated. This makes aspects of learning visible and may provide support for diverse learners. Higgins (2006) cautions, however, that the teachers' subject or pedagogical knowledge may mean that they are unable to make the links between recording and mathematical abstractions.

2.6 The Teaching Model of the NDP

The Strategy Teaching Model (STM) first appeared in *Book 3: Getting started* and was designed by Peter Hughes, although this is not explicitly stated, but, rather, is implicit in the references (Hughes, 2002; Ministry of Education, 2008b).

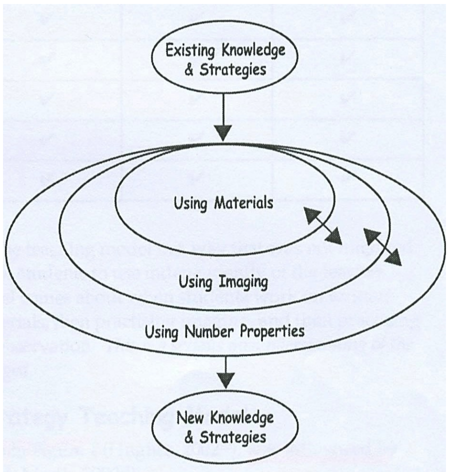


Figure 2.1: The Strategy Teaching Model (Ministry of Education, 2008b, p. 5)

This section now discusses the studies that influenced the development of STM, particularly the work of Pirie and Kieren (1989, 1992, 1994a, 1994b). The STM is described, and its purpose within the teaching programme of the NDP explained. The models' phases of Using Materials and Using Number Properties are discussed in more detail because of their direct relevance to this study.

2.6.1 Background influences on the Strategy Teaching Model

The construction of the STM was influenced by the work of Pirie and Kieren (1989, 1992, 1994a, 1994b; Pirie & Martin, 2000). This influence is acknowledged in *Book 3: Getting started* (Ministry of Education, 2008b) and by Hughes (2002). For this reason, this section gives a brief outline of their research and their theory for the growth of mathematical understanding. A more in-depth discussion of the Pirie and Kieren (P-K) theory can be found in section 2.7.

The P-K theory offers insights into students' learning in any mathematical domain by examining the "cognitive activities of the students as they 'come to know' mathematical concepts" (Pirie, 2002, p. 13). The key elements of the theory are the eight layers of understanding (each containing all the previous understanding) and the concept of 'folding back' to a previous layer of understanding. These elements offer a broad scope in tracing mathematical development (Cobb, 2007). As a student's understanding grows, their thinking can be shown moving forwards and backwards across the layers in a dynamic, non-linear but levelled recursive process (Pirie & Kieren, 1989, 1992, 1994b). The model is characterized by a number of bold rings, called "don't need" boundaries (Pirie & Kieren, 1992). These represent "the ability to operate mentally or symbolically without reference to the meanings of basic concepts or images" (Pirie & Kieren, 1992, p. 248). Figure 2.2 illustrates the Pirie and Kieren's theory for the growth of mathematical understanding.

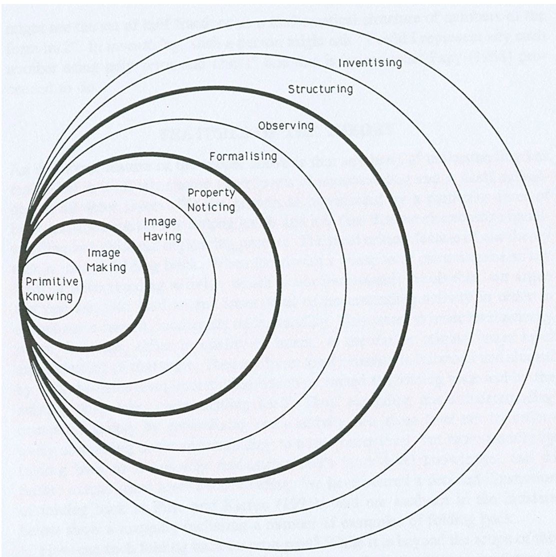


Figure 2.2: Pirie and Kieren's diagrammatic representation of the model for the growth of mathematical understanding (Pirie & Kieren, 1992, p. 246)

Central to the P-K theory is Primitive Knowing, defined by Pirie and Kieren as "the starting place for the growth of any particular mathematical understanding. It is what the observer, the teacher or researcher assumes the person doing the understanding can do initially" (1994b, p. 170). It is the foundation upon which all the later stages of the model are built. Primitive Knowing can be equated to the existing knowledge and strategies phase of the STM. A number of other phases of the P-K theory have influenced the development of the STM. Property Noticing and Formalizing influenced the development of Using Number Properties, and will be discussed in section 2.6.4. Using Imaging was influenced by the Image Making and Image Having stages. These will be discussed in section 2.7, together with the concept of 'folding back.'

2.6.2 Explanation of the Strategy Teaching Model

The Strategy Teaching Model is a teaching tool. Figure 2.1 summarizes the three developmental phases, Using Materials, Using Imaging and Using Number Properties, through which students move as existing strategies are developed into new strategies for each learning outcome in the Number Framework. As students 'move through' each of these phases, their thinking becomes progressively more abstract until they are finally able to manipulate numbers without reference to either images or materials (Ministry of Education, 2008b). The STM provides teachers with a framework (supported by the detailed learning experiences in Books 5, 6 and 7) to use as they introduce and teach learning outcomes.

The teacher's role in supporting student progress through the STM is an important one. All new concepts are introduced Using Materials, no matter what stage of the Number Framework students are working at. Students manipulate materials to solve problems, and, supported by teacher questioning, describe and discuss their actions on the materials. The move to the Using Imaging phase is facilitated by teachers "masking the materials and asking anticipatory questions about actions on those materials" (Ministry of

Education, 2008b, p. 5). Here the role of the teacher is claimed to be critical; "students' progression through Using Materials to the Using Imaging stage of the model is unlikely to be successful without targeted input from you [the teacher]" (Ministry of Education, 2008b, p. 5). Once students can confidently solve problems Using Imaging, teachers encourage the progression to Using Number Properties "by increasing the complexity or size of the numbers involved, thus making reliance on the mental representations difficult or inefficient" (Ministry of Education, 2008b, p. 5).

The concept of 'folding back,' influenced by the P-K model, is another key feature of the STM. The two-way arrows indicate that students are able to move backwards and forwards between the Using Materials, Using Imaging and Using Number Properties phases as they connect actions on materials with mental abstractions.

Although the STM is a key teaching tool, there appears to be very little explanation and elaboration of it for teachers. *Book 3: Getting started* (Ministry of Education, 2008b) provides some detail about how to implement the model. This is supported by the learning experiences outlined in Books 5, 6 and 7, particularly the examples about how to introduce and teach each of the three phases of the STM for specific activities. *Book 3: Getting started* also provides some explanation about the theory behind the phases of the model (particularly Using Imaging) and the differences between the stages of the Number Framework (especially the differences between the counting and part-whole stages). The professional development workshops and in-class support provide further explanation and elaboration of the model. However, in my experience many teachers still lack confidence using the model, particularly the Using Imaging phase. Anecdotally, this can be illustrated by the number of requests I had from teachers to model imaging when I was working as a numeracy adviser. If teachers are not sure of imaging's purpose, how to effectively promote it, or convinced of its merit (as many appear not to be), will they be able to ensure that students have made the necessary connections between both the materials and the image, and (perhaps more significantly) the image and number properties and formal symbolism?

Pirie's (2002) comments about students' ability to connect their image with the formal representation are pertinent when considering the explanations given in *Book 3: Getting started*:

> There is clear evidence from our data (and that of many others) of a disconnected mental leap that many students are required to make as they move from having constructed some mental image(s) of a concept, to its formal representation, without having grasped the relationships and features of their images that enable this abstract generalization of their concrete experiences. (p. 930)

She adds that the transition between levels of understanding is not well understood by the mathematics education community (Pirie, 2002). Perhaps this is why there is so little information in the NDP about how to ensure students understand the connection between their images and the formal representations.

2.6.3 The Using Materials phase in action

The Numeracy Development Project is structured around using materials to introduce and teach mathematical concepts. Many of the materials associated with the NDP have been chosen to foster part-whole thinking. Some are built on familiar materials, such as tens frames and linked cubes, but new materials are also included, for example, the decipipes (see Appendix C). Teachers use materials to introduce new strategy concepts at all stages of the Framework. It is assumed that the materials translate mathematical abstractions into a form that enables learners to relate new knowledge to existing knowledge (Moyer, 2001). The use of materials is an essential part of a progression through the phases of the STM that moves students "from an externalized representation to a visualized idea and then to an internalized representation" (Higgins, 2005, p. 89). Using materials highlights mathematical concepts rather than focusing on the procedural stages used in traditional algorithms (Higgins, 2005).

The teacher generally chooses the materials, based on their purpose for the lesson. The teacher demonstrates, and guides students to use materials to show thinking, and mediates student discussion as they use materials to justify their thinking (Higgins, 2005). Students are supported to see the relationship between their prior and new knowledge, the materials acting as an "integral aspect of the learner's mathematical reasoning rather than an external aid to it" (Anthony & Walshaw, 2007, p. 120; Moyer, 2001). If used in this way, materials become referents which students can use to reason mathematically and communicate their thinking; the materials move from being a working model to a thinking model (Gravemeijer, 1994, as cited in Higgins, 2005).

There are two further important points about the use of materials. First, although students may use and share materials, it is the relationship between their mental actions and the materials that is the basis of the knowledge they acquire (Moyer, 2001). Second, as Higgins (2005) and Moyer (2001) claim, the choice and use of materials may be limited by the teacher's own mathematical content and pedagogical knowledge. This in turn can affect the opportunities students have to discuss mathematical ideas and their ability to think mathematically (Higgins, 2005; Moyer, 2001).

Walls (2004) criticizes the specialized materials used in the NDP for creating artificial representations of numbers. She questions whether these adult-contrived artifacts are as effective for students as everyday objects. In particular, Walls claims that the specialized materials may fail to provide links with numbers as they appear in students' everyday lives (for example, a speedometer rather than a number line), and that students may fail to see the mathematical significance of materials. Walls also argues that the specialized materials may "unintentionally limit [students'] mathematical development," by restricting their exploration of number, "their mathematical 'perceiving' and 'knowing'," and by destroying their belief in "the validity of their own inventions" (Walls, 2004, p. 31). It is important to note that materials do not carry any inherent meaning; rather mathematical relationships must be imposed on them (Moyer, 2001). This may create tensions between the teacher's intended use, and the students' actual use. For example, it is possible that, in manipulating materials, students may construct their own mathematical meaning, which is different to the concept being taught in the lesson (Moyer, 2001). It is also possible that the materials may obscure the purpose of the lesson. There are a number of times where this has happened in my classroom, for example, students focusing on the pictures on the animal strips and describing a number in terms of the number of groupings a particular animal, rather than the numerical groupings of tens and ones. Hughes (2002) justifies the choice of materials associated with the NDP, stating that they are deliberately selected to enable students to readily imagine the materials when they are shielded or removed in the Using Imaging phase.

2.6.4 The phase Using Number Properties

When referring to what are termed 'number properties' by the STM, authors talk about abstractions (for example, von Glasersfeld, 1990), or formal generalizations (Pirie & Kieren, 1994b), However, it is difficult to find a clear definition of exactly what is meant by these terms. Perhaps the clearest is von Glasersfeld's description of abstraction (originally a Piagetian term meaning to abandon materials):

> Mathematics is the result of abstraction from operations on a level on which the sensory or motor materials that provided the occasion for operating is disregarded. In arithmetic this begins with the abstraction of the concept of number from the act of counting. (1992, p. 2)

The Using Number Properties phase of the STM is based on 'Property Noticing' and 'Formalizing' phases of the P-K theory (Hughes 2002; Pirie & Kieren, 1992; see also section 2.6.1). Students who are Property Noticing note connections, combinations and distinctions between images, are able to predict how these might be achieved, and are able to record the relationships. At the Formalizing phase, students take the properties as given; they have a "class-like mental object not dependent on meaningful images" (Pirie & Kieren, 1992, p. 247). The ideas in these two P-K phases led to the construction of the Using Number Properties phase (Hughes, 2002). Students who are Using Number Properties "will abandon the use of materials or imaging…and proceed to reason directly with the numbers and their properties" (Hughes, 2002, p. 354).

In the STM, Using Number Properties is promoted by teachers pushing the "number size up to the point where imaging the numbers is a burden, and solutions are best found by reasoning with abstract number properties" (Hughes 2002, p. 354-355). Hughes illustrates how the size of numbers is increased to make imaging difficult by giving the example of 7 + 89. He comments that the problem would be difficult to solve by imaging, but is solvable using part-whole reasoning, as defined by the STM (Hughes, 2002). However, I suggest that students would be able to create an image to solve this problem, and that their image would provide further support in helping students to reason with numbers and their properties. Discussion now turns to the Using Imaging phase of the STM. This phase is the focus of this study, and the Using Imaging phase is therefore discussed in detail. Studies and international research that influenced its development, and other studies relating to the use of imaging by students, are also discussed.

2.7 The Using Imaging phase

The Using Imaging phase was included in the NDP as a "response to the real problem that many children fail to make the desired abstractions out of the *Using Materials* phase" (Hughes, 2002, p. 353). The difficulty of moving from concrete materials to abstract number concepts is discussed by a number of researchers. Hart illustrated the problem clearly by reporting the comments of an eight-year-old boy, who said that "bricks is bricks and sums is sums." She noted there needed to be a 'bridge' between the bricks and sums (Hart, 1989, as cited in Ministry of Education, 2008b, p. 7). Similarly, Higgins (2005) states that the experiential use of materials does not necessarily lead to students developing mathematical understanding. The need to connect students' own internal representations with the external representations of the materials is highlighted by both von Glasersfeld (1992) and Moyer (2001). Von Glasersfeld emphasizes the difficulty of doing this, saying that, "no matter how trivial and obvious they [the desired abstractions] might seem to the teacher, [they] are never obvious to the novice" (1992, p. 2). Further, abstractions cannot be transferred to the learner; rather they must be developed by the learner themselves (von Glasersfeld, 1990, 1993).

Possible ways of making a connection between materials and abstractions are suggested by Bobis (1996) and Shuhua, Kulm and Wu (2004). Bobis (1996) argues that 'visual imagery' is a powerful tool that helps the students to move from concrete materials to abstract symbols. She contends "that the emphasis be shifted to using visual imagery prior to the introduction of more formal procedures" (Bobis, 1996, p. 21). In a comparative study of the pedagogical knowledge of middle school mathematics teachers in the United States and China, Shuhua, Kulm and Wu (2004) found that, although many US teachers used materials, they did not develop the connection between the manipulative activities and abstract thinking. In contrast, by using examples related to students' lives, materials and concrete models, the Chinese teachers were able to make this connection (Shuhua et al., 2004).

Imaging, as it is used in the STM, is the "creation of picture images." It is a deliberate attempt to introduce the physical imagery of absent objects into teaching and learning (Hughes, 2002, p. 354). Imaging is promoted in two sub-phases (Hughes, 2002). In the first, the teacher shields or screens materials and asks students to either explain how they would manipulate the materials if they were able to touch them, or to explain how the teacher is manipulating the materials. This phase is based on Mathematics Recovery (Wright, 2000). This definition was extended in 2007 to include students viewing but not touching materials, and therefore imaging them (verbal communication from numeracy advisers, August 2007). If students are unable to successfully describe the actions they would perform on the imagined materials, the teacher will 'fold back' (return) to the Using Materials phase to help the student make the connection between the materials and the image they are being asked to create (Ministry of Education, 2008b).

In the second phase, students mentally image materials "without the support of shielding or folding back" (Hughes, 2002, p. 354). If students are successful at this stage, the teacher will move on to Using Number Properties (Hughes, 2002). Although Hughes discusses two sub-phases, and they are used in the teacher resource books (where they are described as 'shielding' and 'imaging only'), only one phase is described in *Book 3: Getting started* – the teacher masking or shielding the materials (Ministry of Education, 2008b). Hughes' statement that students do not have the support of 'folding back' from the second phase appears both to contradict the STM and suggest that there is a hierarchy within the Using Imaging phase. This does not appear to be supported by the STM, which shows students being able to 'fold back' from Using Number Properties to Using Imaging, and indicates only one 'level' within the Using Imaging phase.

2.7.1 The influence of the Pirie and Kieren theory

The influence of the Pirie and Kieren (P-K) theory on the development of the STM has been briefly discussed in section 2.6.1. This section describes the P-K theory in more depth, particularly the Image Making and Image Having phases, and the concept of 'folding back.'

Pirie and Kieren describe their theory as constructivist, "elaborating the nature of understanding as the personal building and re-organization of ones' knowledge structures" (1989, p. 243). Understanding is seen as a dynamic whole, because "one can understand a piece of mathematics in many ways at once" (Pirie & Kieren, 1992, p. 244). The eight levels of the theory move out from the existing knowledge a person brings to a task. This does not imply, however, that the outer levels represent higher-level mathematics. In the same way, Primitive Knowing does not imply low-level mathematics; primitive is taken to mean "'prime' as in both 'important and previous'" (Pirie & Martin, 2000, p. 129). Primitive Knowing is the starting point for the growth of any particular understanding. It is the point that the observer (usually a teacher) assumes is in the students' heads when beginning a new activity (Pirie & Kieren, 1989, 1992, 1994b). The theory is described as recursive because "thinking moves between levels of sophistication…each level is in some way defined in terms of itself (self-reference, self-similar), yet each level is not the same as the previous level" (Pirie & Kieren, 1989, p. 8). Another key feature of the P-K theory is the dynamic movement within and between the eight levels in order to grow understanding (Pirie & Kieren, 1992, 1994b).

The Image Making and Image Having phases of the P-K theory have influenced the development of the STM (Hughes, 2002; Ministry of Education, 2008b). Hughes also claims that the meaning of 'image' is far more complex in the P-K theory than the creation of picture images advocated in the STM. For this reason, Using Imaging is "only tenuously linked to the P-K theory" (2002, p. 354). The Image Making and Image Having phases of the P-K theory will now be discussed.

At the level of Image Making, the student begins to form images based on their Primitive Knowing to get an idea of what the concept is about. The images can be visual and pictorial, or they can include ideas expressed in either language or action (Martin, 2008). Pirie and Kieren's (1989, 1992, 1994a, 1994b) discussions of students' problem-solving strategies include images created by folding sheets of paper to represent fractions, drawing tables or graphs to represent quadratic functions, manipulating fractions shapes and drawing diagrams or pictures. The activities at the stage are singular and directed (Pirie & Kieren, 1992).

Pirie and Kieren (1994a) describe students working at the Image Having stage as "working with metaphors. For them, mathematics *is* the image that they have and their working with that image" (p. 40). The students take the image as given; they do not need to recreate it every time it is used (Cobb, 2007). They "carry a mental plan for [the] activities with them and use it accordingly" (Martin, 2008, p. 65). Pirie and Kieren (1994a) describe an example of the progression from Image Making to Image Having. After experiences creating fractional parts by folding a sheet of paper (the unit) (Image Making), a student creates a specific and context dependent image (Image Having). They describe this image as, "'eights [sic] are the pieces I get when I make three folds'" (Pirie & Kieren, 1994a, p. 40). This student is at the Image Having stage. The image itself can be used to solve problems; the student no longer needs to perform particular physical actions to obtain an image (Pirie & Kieren, 1989, 1994b).

Image Having is the first level of abstraction. Images are no longer tied to actions (as in Image Making); rather they are "replaced by a form for the images" (Pirie & Kieren, 1994a, p. 8). The "person's mathematics is freed from the need to perform particular actions. The image itself, as a mental object, can be used in mathematical knowing" (Pirie & Kieren, 1992, p. 247). Understanding develops further as students notice properties of their images: noting distinctions, combinations or connections between images (Pirie & Kieren, 1989). In the STM, this would represent the beginning of the transition between Using Imaging and Using Number Properties.

Another key feature of the P-K theory that influenced the development of the STM is the notion of 'folding back' (Hughes, 2002; Ministry of Education, 2008b). The P-K theory recognizes that the growth of understanding is not a smooth outward movement, but a continual movement back and forth between the layers (Martin, 2008). Students 'fold back' (return) to a previous level if a problem is not immediately solvable and to extend the level of understanding which they have found is currently inadequate (Pirie & Kieren, 1992). In 'folding back,' students do not return to their original inner level actions; rather, students retrieve and build on their "thicker understanding at the inner layer to support and extend their understanding at the outer layer that they subsequently return to" (Pirie & Martin, 2000, p. 131). 'Folding back' is vital to the growth of understanding and "reveals the non-unidirectional nature of coming to understand mathematics" (Pirie & Kieren, 1994b, p. 174).

As students work to solve problems, they move in different ways and at different speeds through the P-K model to develop understanding (Pirie & Kieren, 1994b). A number of examples of the different pathways taken by students are described by Pirie and Kieren. For example, one student 'folded back' from Observing to Image Making to develop his understanding of quadratic functions. Another 'folded back' from Formalizing to Image Making to understand the additive properties of fractions (Pirie & Kieren, 1989, 1994b).

The description of 'folding back' in the STM highlights the influence of the P-K theory: "folding back to previous phases of the model is critical as students attempt to connect the mathematical abstractions with actions on materials" (Ministry of Education, 2008b, p. 5). There appears, however, to be a significant difference between the use of 'folding back' in the STM and the P-K theory. The P-K theory emphasizes the unique path each student takes. In contrast, the STM suggests a linear path, with students moving backwards and forwards between Using Materials and Using Imaging or between Using Imaging and Using Number Properties (Ministry of Education, 2008b). My perception is that the pathways students follow are fluid, and closer to those described by the P-K theory than in the STM.

The P-K theory offers a theoretical way of looking at the growth of understanding as it happens, and attempts to elaborate on the constructivist definition of understanding in which the learner continually reorganizes their knowledge (Pirie & Kieren, 1994b; Pirie & Martin, 2000). In considering the theory and its implications for teaching and learning it is "important to realize that these levels and features…exist in the domain of the observer" (Pirie & Kieren, 1992, p. 245). The levels were developed by watching people doing mathematics, and validated because features of people's behaviour were found to fit the theoretical model (Pirie & Kieren, 1992). It is an explanatory model of learning rather than an instructional model.

In the last twenty years, a number of other studies have been conducted into students' use of imaging. Although most of these did not influence the development of the STM, they are relevant to the focus of this study, and so are discussed below.

2.7.2 Further research related to imaging

Gray and Pitta's (1996, 1997, 1999a, 1999b) studies focus on investigating low- and high-achieving students' mental processes as they solved the same problems. They found that there is a difference in the types of images reported by the two groups of students. Low-achieving students described seeing and manipulating images of concrete materials such as die, counters, marbles, fingers or a number track, and sometimes also described the colour (Gray & Pitta, 1999b). Gray and Pitta describe these images as analogical representations of physical objects. Gray and Pitta (1999b) comment that, although using mental images, the actions these students performed were reminiscent of the activities they would have performed on real objects. Almost all the actions involved counting, rather than deriving facts and using what they knew about numbers (Gray & Pitta, 1996). They conclude, the "objects and the actions are *essential for thought* in that they guide the use of a procedure which may or may not be successful" (Gray & Pitta, 1999b, p. 14).

In contrast, the images created by high achievers appeared to act as thought generators and memory aids, coming to the fore momentarily to enable new actions or transformations to take place (1999b; Gray, Pitta, & Tall, 2000). Rather than focusing on their actions on objects, they were "able to focus more flexibly on the results of those actions expressed as number concepts" (Gray et al., 2000, p. 411).

The different types of image also appear to have an impact on students' working memories. Gray, Pinto, Pitta, and Tall (1999) found that the mental manipulation involved in the procedural activities of low-achieving students appears to strain their working memory. Low-achieving students appear to "show an *extraordinary* use of working memory," because they are unable to filter out unnecessary information (Gray et al., 2000, p. 410). High-achieving students appeared able to focus on relevant detail at the appropriate moment, to make choices, and filter out irrelevant information, thus making much more efficient use of their working memory (Gray et al., 2000).

The low-achieving students' images reported by Gray and Pitta appear to be similar to the picture images described in the STM. The use of these is "promoted by the teacher masking materials and asking anticipatory questions about actions on those materials" (Ministry of Education, 2008b, p. 5). The resource books support students' use of picture images closely tied to actions on the materials. Two examples illustrate this. The Using Imaging section of *Pipe music with decimals* suggests that students image what the models of decipipes might look like (Ministry of Education, 2008c). The Using Imaging example for *Birthday cakes* is even more closely linked to actions on materials. It is suggested that teachers show students one-fifth of a paper circle with four counters on it, after which they ask the following question: "Here is a piece of Rongopai's birthday cake. Each piece of the cake has the same number of candles. How old is Rongopai?" (Ministry of Education, 2008c, p. 27).

Findings from Bobis' 1996 study are discussed as part of the theoretical background to the STM (Ministry of Education, 2008b). Bobis studied how kindergarten children used visualization strategies to mentally combine and separate numbers represented by arrangements of dots. Children were encouraged to verbalize the patterns they saw for different numbers, and to manipulate their mental images by adding or subtracting dots. Children also explained how they made their new number using drawings, concrete materials and verbal descriptions. These findings indicate the benefit of visualization to young children's development of number sense. The children in Bobis' study were mentally able to combine and separate numbers, which enabled them to instantly recognize the whole and its related parts.

2.8 Aims of this study

A great deal is still unknown about imaging, both about how it is used by students as they solve problems, and how imaging helps them to develop abstract number concepts. Two classroom number units, taught using the progressions of the STM and the NDP resource materials in *Book 5: Teaching addition, subtraction, and place value* and *Book 7: Teaching fractions, decimals, and percentages*, provide the context for investigating the research questions, and for tentative conclusions to be drawn about how imaging supports the development of abstract number concepts.

My research questions are:

- What images do students create when solving mathematical problems?
- When do students image as they solve mathematical problems?
- How do students use imaging when solving mathematical problems?
- How does imaging influence students' abilities to solve increasingly difficult problems?

The nature of the images generated by students, the role imaging plays in students' developing mathematical reasoning, and the interrelationship between imaging and materials and number properties are central to this study.

The study also focuses on the complexity of students' thinking as they solve problems. Imaging appears to be a crucial transition point, linking actions on materials with the abstract numbers of Using Number Properties. We are told that the "gap between material and abstraction is often too great for students to bridge," and that producing mental images of absent or shielded materials is a way of creating a 'bridge' (Ministry of Education, 2008b, p. 7). Despite this, we appear to know very little about how students use imaging. Further, it appears that some teachers may be using the phases of the STM (particularly imaging) without really understanding how to support students' developing understanding. If teachers do not understand how and why they are teaching the Using Imaging phase, it is unlikely that they will successfully make the connections students need between the materials, images and number properties. It is also unlikely that students will understand the importance of imaging, particularly when faced with a difficult problem.

I have decided to define imaging as 'the mental and drawn representations used by students as part of the mathematical process to solve problems,' and have separated imaging into 'picture images' and 'transformed images.'

Picture mental and drawn images: these are direct copies of the materials used to make and solve the problem during the teacher-led Using Materials phase. Students manipulate these images in a way that replicates the process modelled by the teacher to manipulate concrete materials.

- *Mental picture image*: the image is imagined in the students' heads. Students create mental picture images when they imagine the materials that were used to make and solve a problem. Mental picture images are also created when students describe how they would manipulate materials that could be used to solve a problem, but that have been shielded so they are not visible. A final way in which mental picture images are created is when students describe how they would manipulate materials that are visible but that they are unable to touch. These images are similar to those used in the Using Imaging phase of the Strategy Teaching Model (Ministry of Education, 2008b), and the 'picture images' described by Hughes (2002).
- *Drawn picture image*: students draw a representation of the materials used to make the problem (for example, a circle to represent the paper circles used to model birthday cakes, and dots to represent the candles).

Transformed mental and drawn images: these are images generated by students to support solving problems. These images are not linked to the materials used by the teacher or students to make the problem during the Using Materials phase. In addition, transformed images are often generic images that can be used to solve a range of problems, for example an empty number line. Transformed images can either be mental (imagined) or drawn.

Other definitions relevant for my study are:

Materials: the concrete materials used to make and solve a problem. These are usually selected by the teacher to meet the needs of students and the purpose of the learning outcome, but may also be chosen by students from a range of appropriate materials.

Number properties: students who are using number properties are able to solve problems (with understanding) by manipulating just the numbers, and with no reference to any type of image or concrete materials.

Discussion of the concept of imaging raises a number of issues, as well as questions. It is suggested that imaging is a thinking process used by students (and promoted by teachers) to link knowledge gained by manipulating concrete materials with number properties. Do all students create images? What are the images created by students? Is imaging a 'stage,' as seems to be implied by the use of the 'Using Imaging phase' in the STM, through which students pass through and move beyond. Do all student use the imaging 'phase' in this way, or do some need to continue to create images to make sense of the problem? If imaging is a phase or stage, how do students move from imaging to number properties? Is it a direct move or is there something in between? How do we ensure that students have made the connections between their images and the abstract numbers that will enable them to solve problems without reference to either an image or materials? I will return to these questions and their implications for teaching and learning in Chapters 6 and 7.

2.9 Summary

This chapter has explored the theoretical foundations behind mathematics teaching and learning in New Zealand, and the origins and key features of the Numeracy Development Project, and the Strategy Teaching Model. It has focused, in particular, on the Using Imaging phase of the STM, the influence of the Pirie and Kieren theory on the development of the STM, and on research into the role of imaging in helping students to acquire abstract number properties. Chapter 3 describes the methodology and research design for this study.

Chapter 3: Methodology and methods

3.1 Introduction

This chapter describes the methodology and research design for this study. Section 3.2 discusses the interpretive methodology used and case studies. Section 3.3 describes the research design, including the context of the study, the role of the researcher, the participants, and the teaching programme. The data collection methods discussed in section 3.4 include observation, interviews, field notes and documents. Section 3.5 discusses ethical considerations. Issues relating to trustworthiness are considered in section 3.6, including changes to the research design. The chapter concludes with a description of data analysis process and the five main analysis themes, and related sub-themes that emerged from that process.

3.2 Methodology

The main purpose of this study is to more closely examine the concept of imaging within mathematics lessons from the perspectives of two groups of students. Because I am researching students' activities in the context of the classroom certain methodologies are more relevant. My study involves two case studies, and I have chosen to use an interpretive methodology.

3.2.1 Interpretive studies

In order to study the role of imaging through the 'eyes' of students my study uses a naturalistic, interpretive approach (Cohen, Manion, & Morrison, 2007). Neuman (2000) defines the interpretive approach as:

> the systematic analysis of socially meaningful action through the direct detailed observation of people in natural settings in order to arrive at understandings and interpretations of how people create and maintain their social worlds. (p. 71)

Interpretive methodology is characterized by concern for the individuals and the processes being studied. The researcher must use various methods to develop an understanding of the individuals being studied, to find out as much as possible about what and how they are doing and thinking (Cohen et al., 2007). In order to develop this understanding, I will need to 'bracket' the term imaging, a term taken for granted by students and teachers, before questioning and re-examining students' experiences (Bogdan & Biklen, 2007; Neuman, 2000). By doing this I hope to make "visible the underlying scaffolding of understandings on which actions are based" (Neuman, 2000, p. 75).

The research is naturalistic because it "observes ordinary events in natural settings" (Neuman, 2000, p. 349). The study is set in the customary environment of the participants (a classroom, although not their regular classroom). The context of a mathematics teaching unit is a naturally occurring situation (Cohen et al., 2007; Davidson & Tolich, 2003). By investigating what is taking place in situ, I hope to obtain more authentic data (Cohen et al., 2007). This will enable me to develop an understanding of the role of imaging from the viewpoint of the students (Lichtman, 2011), and ensure their experience is distorted as little as possible (Bogdan & Biklen, 2007).

Interpretive researchers often begin with individuals and set out to understand how they interpret the world around them, in this case how students use imaging to help them solve problems. Researchers use interpretation to theorize. This theory is emergent, arising from a particular situation, and is 'grounded' in the research data, and theorizing follows rather than precedes the data (Cohen et al., 2007).

3.2.2 Case studies

Merriam defines case studies as "an examination of a specific phenomenon, such as a program, an event, a person, a process, an institution, or a social group" (1988, p. 9, as cited in Lichtman, 2011, p. 111). Case studies are a comprehensive research strategy which are strong on reality (Cohen et al., 2007; Yin, 1994). They enable the researcher to take account of contextual conditions pertinent to the study, which, in this study, included the people (the students and the teacher), the school, and the mathematical teaching units. Further, case studies are adaptable, enabling the researcher to investigate "unique example[s] of real people in real situations," while allowing for unanticipated and uncontrolled variables (Cohen et al., 2007, p. 253). In this way case studies develop the case's "own issues, contexts, and interpretations, its *thick description*" (Stake, 2003, p. 140). Case studies can have a small focus, but at the same time deal with a full variety of evidence, for example, documents, observations and interviews (Yin, 1994).

There are a number of disadvantages to conducting a case study. They take time, and the contexts can overwhelm the data. Another concern is that case studies are not generalizable. Yin (1994), however, disagrees with this, arguing that it is possible to generalize case studies to theoretical propositions. Another disadvantage is that the researcher may become too close to the context and issues involved in the study.

The study discussed here consists of two associated case studies. Both investigated the same mathematics programme within the same setting, but with two sets of activities and two groups of people. Both studies had a small focus, and both aimed to "illuminate, support, or challenge previously held assumptions" by presenting "thick description" (Lichtman, 2011, p. 111). The design of the study will be discussed in more depth in the following section.

3.3 Research design

3.3.1 Context

This study took place in a large full primary school (Year 0-8) over a six-week period in Term 2 of 2011. It involved teaching mathematics units to two groups of Year 6 students (10-11 year olds). These units were based on the school's mathematics programme, which included the Numeracy Development Project material.

3.3.2 Role of the researcher

I have been a teacher and team leader at the school for three years, and had taught all but one of the students in 2010. In 2011 I was on study leave, researching and writing this thesis as part of my Master of Education. I was both the teacher and the researcher during this study, often known as insider research. A unique aspect of insider research is that the researcher undertakes the research in a setting where they already have relationships, and where they have access to inside knowledge about the school and students (Anderson, Herr, & Nihlen, 2007). A consequence of this was that I had credibility with both the students and their parents, and did not have to spend time establishing a rapport or gaining their trust and confidence (Mutch, 2005).

There is a sense of ownership and control in taking the role of the teacher researcher because I can choose to examine knowledge and teaching practice that will benefit both the students and my own practice. I am able to conduct the research at my own pace and in a direction I choose in response to the data collected (Gregson, 2004). In doing so I have to be sensitive to what is happening in the field, and to acknowledge that the research involves social relationships and personal feelings, both mine and the students in the group (Neuman, 2000).

There are a number of limitations to insider research, mostly because the researcher has a dual purpose. On one hand, their role is to engage in activities appropriate to a situation, while, on the other, observing the activities, people and physical aspects of that situation (Anderson et al., 2007). In fulfilling this dual role I have to ensure that my research is ethical and respectful.

> It also needs to be humble. It needs to be humble because the researcher belongs to the community as a member with a different set of roles and relationships, status and position. (Tuhiwai Smith, 1999, p. 139)

I also need to be mindful of any effects of my research findings on other members of the school community such as teachers and principals, particularly if the findings challenge teaching theories or practices (Gregson, 2004).

Balancing the role of teacher and researcher can also be difficult due to the need to balance the collecting, collating and reviewing of data on top of a normal teaching load (Gregson, 2004). I was lucky not to face this dilemma because study leave meant I did not have my usual teaching commitments. However, not having my own class limited the data collection period because I had arranged to work with the students for a finite amount of time. This meant I would not have the usual flexibility of an insider researcher to extend the data-gathering period to add to the range and depth of data gathered.

Taylor and Bodgan (1998) comment that a researcher should avoid any activity that interferes with the ability to collect data. As an insider researcher this is difficult because of the need to balance data collection with being responsive to individual needs (Gregson, 2004). Because I am responsible for ensuring that the best possible teaching and learning takes place for all students, I have to acknowledge that this might not be compatible with an objective data collection process.

As researcher, I am the research instrument. As such I have to ensure I constantly analyse and acknowledge my own feelings and subjective reactions. In particular, I have to view data in a reflective and objective way, from the perspective of a stranger (Anderson et al., 2007).

3.3.3 Participants

The participants in this study were drawn from two existing mathematics groups, nine students in group 1 (five boys and four girls) and seven in group 2 (two boys and five girls). The classroom teachers decided the exact composition of each group. I asked to work with these two groups because they had (with the exception of one student) been part of my mathematics class in 2010. This meant I already had a relationship and rapport with the students (Cohen et al., 2007). Students are more likely to respond openly and honestly if they feel respected and safe, and the skill of the researcher is to put them at ease, and establish shared interests and dialogue (Smith, 2011).

I chose a sample of ten from within these two groups (seven from group 1 and three from group 2) in order to represent the range of ability within each group and a variety of approaches towards solving problems. This was therefore a purposive sample, where cases are handpicked on the basis of the researcher's judgement about their typicality or because they possess the particular characteristics being sought (Cohen et al., 2007).

3.3.4 Teaching programme

Classroom teaching

The teaching units developed achievement objectives from the number and algebra strand of the mathematics learning area of the *New Zealand Curriculum* (*NZC*) (Ministry of Education, 2007a), and utilized the teaching programme outlined in the Numeracy Development Projects (Ministry of Education, 2007b, 2008b). Group 1 was working at the beginning of Level 4 of the *NZC*, which equates to Stage 7 (Advanced Multiplicative) of the NDP. They had not yet started Stage 7, and so their unit introduced the addition and subtraction of decimal fractions. Group 2 was working at Stage 5 (Early Additive), which equates to Level 2 of the *NZC*. At the beginning of the study, group 2 students were half way through a six-week block of numeracy teaching, and I continued their addition and subtraction topic. I used the diagnostic information gathered in the initial interviews to inform my planning for this group.

The planning for both groups was flexible. I planned a starting point for each unit, but had no fixed end point; the time spent on each learning outcome varied across each group. I planned two or three lessons at a time, but often modified and revised the programme after each day's teaching. Appendix D gives details of the main content of each lesson, and the key activities, materials and resources used.

Each new strategy was introduced using the progression outlined in the Numeracy Development Projects Strategy Teaching Model (STM) (Ministry of Education, 2008b). Learning outcomes and exemplar problems were recorded in each group's modelling book. Strategies were introduced Using Materials. Students moved from Using Materials to Using Imaging and, finally, Using Number Properties. Progress, however, varied between members of the group, and depended on when they (or I) felt confident to move to the next phase. Most lessons included a combination of teacher-led and independent activities, although the exact balance varied across lessons, usually depending on the complexity of the strategy being taught.

Students worked initially with materials to solve problems that were presented within realistic contexts. Materials were also available for students to use at all times during the teaching units. Students were encouraged to draw and record their thinking, and to use whichever phases of the STM helped them successfully solve the problem.

Discussion was an important aspect of all lessons. I led discussions to decode and interpret learning outcomes and problems, particularly the key words, in order to ensure all members of the groups understood what they were being asked to do. Questioning was used to support developing knowledge, and to encourage discussion between students. For example, students were encouraged to explain and justify their chosen strategy to other group members, including how they had used materials or imaging to help them solve problems. A sample list of numeracy questions can be found in Appendix E. Questioning and group discussions also gave me the opportunity to clarify thinking and to model appropriate mathematical language by paraphrasing students' responses. Think-pair-share (see section 2.5.3) was a key component of discussions, particularly during teacher-led activities. During both teaching units, students were asked to share solutions with, and explain their thinking to another person, before sharing them with the group. Initially, these pairings were fluid and depended on whom each student was sitting next to. However, after observing students' learning needs during the first few days of each unit, I asked specific students to work together. Students' solutions were recorded in the modelling book. This led to the sharing of a variety of strategies for solving a particular problem, and to students critically evaluating each other's strategies.

Assessment during the teaching units was based on observation of and conversations with students. This mostly consisted specific feedback and feedforward of the each student's next learning step. This anecdotal data also informed my planning. A formal assessment of mathematical learning took place at the end of each teaching unit. It consisted of a semi-structured interview with each student, based on the school's 'I can' numeracy assessment programme which had been modified to enable me to assess all the learning outcomes taught. Students were asked to attempt to solve each problem using mental strategies, and to explain their thinking and strategy. Paper and pencils were available if they wanted to record their thinking.

Resources

The teaching resources were based on the NDP materials, particularly *Book 5: Teaching addition, subtraction, and place value* and *Book 7: Teaching fractions, decimals, and percentages* (Ministry of Education, 2007c, 2008c) These were supplemented by activities from the *Figure it out* series (Ministry of Education, 2002-2005), *New Zealand Curriculum Maths, Stage 7, Book 2* (Tipler & Timperley, 2007), the Assessment Resource Banks (www.nzcer.org.nz), and problem-solving activities from nzmaths (www.nzmaths.co.nz). Full details of the resources used for each lesson can be found in Appendix D. The materials used to model strategies were those suggested in Books 5 and 7. Illustrations of the materials used can be found in Appendix C. Although specific materials were used to model different learning outcomes, students were able to use their preferred materials when solving problems.

Setting

The two groups in the study were ability-based groups. The classroom teachers determined the composition of each group, based on assessment data gathered using the school-wide 'I can' assessment at the end of 2010 and in Term 1 of 2011. There was a range of ability within both groups, but particularly within group 1.

The data collection took place over six weeks in May, June and July. The beginning of group 2's unit was delayed by a week due to the closure of the school after the 13 June earthquake. In Year 6 mathematics is taught four times a week (Tuesday to Friday) for an hour between 9.30 and 10.30 a.m. The lessons took place in a classroom that was to be used as a junior classroom the following term – furniture and equipment were already in the classroom, although generally pushed to one side. This limited the space available for teaching, and led to some complaints about the chairs being too small when the groups worked at tables!

Most group 1 students were there every day, although there were several occasions when one or two students were over 15 minutes late. One student in group 2 worked with a teacher-aide for the first 30 minutes of each mathematics lesson, and joined the group after that. Fridays were disrupted by music lessons, with three group 2 students having lessons (at different times) between 9.30 and 10.30 a.m. On one occasion group 2's lesson had to be cut short because of a Year 6-wide spelling assessment, and on two other occasions we had to find another teaching space because the classroom was unavailable.

3.4 Data collection methods

I was a participant observer, taking an insider role in the group I was studying, and engaging with the activities I set out to observe in order to obtain detailed information about what was happening (Cohen et al., 2007). Taylor and Bogdan describe a participant observer as "becoming an unobtrusive part of the scene, people the participants take for granted" (1998, p. 45). A participant observer has the advantage of being able to see events evolve over time, and to catch the dynamics of the situation and the people (Cohen et al., 2007). It is also suggested that the data are both richer and deeper because of the presence of the teacher (Gregson, 2004). Participant observation is a method that relies on "watching, listening, asking questions and collecting things" (LeCompte & Preissle, 1993, p. 196). It is usually combined with other means of gathering data including interviews and document collection (LeCompte & Preissle, 1993). This was the situation in my study. Table 3.1 gives details of the data collected during the study.

Data sources	Group 1	Group 2	Total
Video recordings	11 tapes – sequential lessons (1 missing – 26/5)	13 tapes – sequential lessons	24
Edited video recordings	10 lessons: truncated versions of important sections of each day	12 lessons: truncated versions of important sections of each day	22
Field notes	10 lessons: field notes recorded after watching video recordings of each lesson Assorted, dated anecdotal notes	13 lessons: field notes recorded after watching video recordings of each lesson Assorted, dated anecdotal notes	23 lessons Assorted
Video-cued interview transcripts	9 (transcribed)	5 (transcribed)	14
Audio recordings	3 (transcribed)		3
Students' work	Samples for 8 children – dated	Samples for 6 children – dated	14
Pre-assessment interviews	9 transcribed interviews	6 transcribed interviews	15
Post-assessment interviews	9 transcribed interviews	6 transcribed interviews	15
Modelling book	Learning outcomes and examples of problems and solutions - dated	Learning outcomes and examples of problems and solutions - dated	

Table 3.1 Record of data collected

3.4.1 Methods of observation

I used a video camera to record observations of all teaching sessions. This decision was informed by my 2007 small-scale investigation, during which I had recorded observations using field notes. I knew it would be very difficult to record detailed written observations while teaching a large group, and thus balance my dual role of teacher and observer. The camera provided a practical means of recording data. It was located in a fixed position to one side and slightly in front of the groups, chosen because it was a relatively unobtrusive place to put the tripod, but enabled me to see and hear the groups working (Anderson et al., 2007; Creswell, 2005). The camera captured everything within its focus; the data was not presented from my point of view because I was not deciding what to record and, therefore, what to leave out, or emphasizing an event by zooming in on it (Robson, 2011). It is often suggested that two cameras are used to record lessons, one to capture the students and the other the teacher (Anthony, 1994; O'Brien, 1993). However, I used a single camera to record lessons. I do not feel this detracted from the images collected because I was able to capture both what I was doing and the students in a single frame.

An advantage of using the video camera was it allowed me to balance the insider-outsider dichotomy of being a participant observer. Neuman (2000) comments that loss of perspective is one of the criticisms of research by a participant observer. The camera, however, allowed me to "take off [my] participant [hat], put on [my] researcher [hat] and analyze the data at [my] leisure" (Anderson et al., 2007, p. 200). I did not have to interrupt the flow of the lessons to record notes, or try to remember what had been said. This made the teaching and learning environment less artificial for both the teacher and students.

A key feature of the research was to record how students reacted to materials, and what they did when they were solving problems, as well as what they said. The video camera allowed me to capture the multimodal nature of the classroom, including action, body language, facial expressions and verbal interactions (Otrel-Cass, Cowie, & Maguire, 2010; Robson, 2011). It provided a lasting visual and audio record of what had occurred, and, as 'thick data,' preserved the dynamics of complex interactions, captured subtle interchanges, and the unexpected (Otrel-Cass et al., 2010; Taylor & Bogdan, 1998). This helped me to see and understand students' thinking processes as well as hearing their voices (Robson, 2011).

There are a number of disadvantages related to using a video camera in the classroom. The video captures only what is in the frame of the lens (Otrel-Cass et al., 2010). Although I positioned the camera so the whole group could be seen as they worked either in a circle on the floor or at tables, it did mean I saw the backs of the students closest to the camera rather than their faces. The constraints of the room meant I was unable to position it directly behind where I sat which would have enabled me to see all students' faces. It also meant it was easier to hear the voices of students closest to the camera. A second disadvantage occurred when students were working independently (often in different parts of the room) or a number of students were talking at once when sharing ideas with their partners. At these times it was often difficult to hear what they said. This meant that I had to rely on capturing their thinking as they fed their ideas back to the group, or by referring to the notes I made as I moved around the classroom, talking to and working with individuals or groups. I also used a digital audio recorder on three occasions during week 3 of the decimals unit when the group was working independently in different places around the room. The audio recorder allowed me to capture in-depth conversations I was not able to observe and I knew would not be captured in sufficient detail by the video camera.

3.4.2 Interviews

An interview is a "construction site of knowledge. An interview is literally an *inter view*, an inter-change of views between two persons conversing about a theme of mutual interest" (Kvale & Brinkmann, 2009, p. 2). Consequently, interviews are frequently used in educational research. There are a number of advantages and disadvantages to the use of interviewing. One advantage is that interviewing is the "process of getting words to fly" (Glesne, 1998, p. 67). Interviews provide information you cannot observe directly, and data that directly addresses the questions asked in the study (Creswell, 2005; LeCompte & Preissle, 1993). In addition, the interviewer has more control over the type of information received because they are able to guide the questions (LeCompte & Preissle, 1993). The disadvantages of interviews include the possibility that the presence of the interviewer will affect the interviewee, that the interviewee may provide information from the perspective they think the interviewer wants to hear, and that interviews do not take place in natural settings (Cohen et al., 2007; Creswell, 2005). Added to this is a concern that the importance of the interviewer may be magnified because they are the instrument for obtaining the knowledge (Kvale & Brinkmann, 2009).

I used two types of interviews during the data collection process: semi-structured and stimulated-recall interviews.

Semi-structured interviews

I conducted two semi-structured interviews with each student, one at the beginning of each group's unit and one at the end. In the initial interview students were asked to solve a series of problems that involved strategies that had previously been taught. Group 1's questions related to learning outcomes I had taught the previous year when working with the group. The questions I asked group 2 related to learning outcomes taught during Term 1 of 2011, and were devised after consulting the group's teacher. The questions were chosen in the expectation of getting a range of responses from the students, but also to ensure that students would not find the problems threatening because they were strategies they had been previously taught. The final interviews asked students to solve problems that had been part of the teaching units during the study, thus providing summative assessment data. Both interviews enabled me to gather descriptive data about the strategies students used to solve problems, and, in particular, the strategies used if a problem was difficult for them. I recorded the interviews (using either a digital voice recorder or video camera) to provide as complete as possible record of what had been said (Glesne, 1998). I also took notes, writing down their answers to the problems and key aspects of their explanations.

Both interviews were guided by a set of key questions (see Appendix E). The advantage of these was that I was confident I would get comparable data across all the students, while allowing their voice and ideas to come through (Anderson et al., 2007; Bogdan & Biklen, 2007; Mutch, 2005). The relaxed structure of the interviews also enabled the students to take whatever direction they chose, and gave me the opportunity to hear in detail their descriptions of the strategies they chose to use to solve the problems (Anderson et al., 2007; Janesick, 2003). I also had the flexibility to react to what they said and to probe in order to seek clarification, more information or description, or to ask students' to evaluate their strategies (Bogdan & Biklen, 2007; Cohen et al., 2007; Glesne, 1998; Patton, 2001). This was particularly important because, as a participant researcher, I shared the experience of numeracy teaching with the students, and had to be careful not to assume that I knew what the student was saying, but, rather, to seek elaboration when appropriate (Anderson et al., 2007). Anderson et al. (2007) comment that it can be challenging to construct interview questions to ensure they are clear and effective in terms of exploring the area of interest. The questions I asked were similar to those used in numeracy lessons, which meant the students were familiar with them. I had also successfully used similar questions in my 2007 small-scale study.

In interviewing the students I had to be careful about how far to go in my questioning (Kvale & Brinkmann, 2009). There were a number of occasions where students found the questions difficult. The students' facial expressions or hesitation also suggested that a number were worried about giving the wrong answer. In these instances, I suggested we moved on to the next question, and assured them that they had helped me by giving me information about how they were trying to solve the problem. I was also aware of the need to reassure students that the initial interview was not a test, but a way of me finding out how they worked to solve problems. Another disadvantage was the time needed to complete the interviews. This was particularly relevant for the final interviews when I had to work around other classroom and school commitments. This meant that the length of some of the interviews was constrained or they were somewhat hurried (Anderson et al., 2007).

Stimulated-recall interviews

Stimulated-recall interviews often involve showing students video clips of themselves taking part in activities or discussions and asking them to talk about what they were doing or thinking. Nuthall (2000) comments that using a video clip is a more powerful cue for recalling a past event than just a standard interview. This is supported by Anthony, who claims that stimulated-recall interviews provide "a visual record of overt learning behaviours, as well as access to students' thoughts and covert behaviours" (1994, p. 128). While

Anthony's research involved secondary aged students, Robson (2011) found that younger children paid close attention to things they comprehend, which means it is possible to infer that they are likely to comprehend an image of themselves and might recall the circumstances under which the images were recorded. I used semi-structured stimulated-recall interviews to supplement the data gained through observation and to obtain retrospective accounts of students' thoughts and actions as they tackled a variety of problems (see Appendix E for examples of the questions asked). These interviews took place outside the normal teaching sessions, but within the classroom we were using. The interviews were recorded to ensure participants' ideas were comprehensively recorded, and I also wrote additional field notes.

Students were shown a variety of clips (ranging from three to seven in a single interview) taken over a number of days. Examples from their books and or the group modelling book were sometimes used, if related to a video clip, to clarify their thinking or further stimulate their memory. On one or two occasions, transcribed extracts of audio recordings were also included so I could obtain more detail or clarify what students had been thinking as they solved problems. I selected the video clips for the stimulated-recall interviews, and prioritized the examples to be shared, including where the replay began and ended, and therefore which part of the sequences would be highlighted. It is possible that this meant that the clips I showed did not accurately reflect students' thinking and strategies for solving problems. I feel, however, that my impact as an observer was lessened by the variety of clips shown to each student and by the fact I collected large amounts of data over a reasonable period of time (Robson, 2011).

An advantage of using stimulated-recall interviews is that students provide rich data and are able to discuss not only their actions during the lesson, but to justify their choices and ideas (Anthony, 1994). Anthony (1994) found that data gathered using this method reflects students' ability to discuss their learning, and that students who are less articulate may find this process difficult. I found there was a contrast between the two groups; students in group 1 were articulate and enjoyed watching the clips and talking about what they had been doing, while the students in group 2 often found it more difficult to describe and explain what they had been doing. There were occasions when students in both groups found it difficult to recall what had happened or why they had chosen to use a particular way of solving a problem. When this happened I was often able to prompt their recall by filling in details about the context. Anthony (1994) notes that a limitation of stimulated-recall interviews is that the reported strategies only represent a sample of those used. I showed students multiple clips spread over a number of days and representing a range of problems and contexts, which may have led to a greater variety of reported strategies.

3.4.3 Field notes

The successful outcome of a participant observation study often depends on detailed, accurate and extensive field notes (Bodgan & Biklen, 2007). These can provide a "personal log that helps the researcher to keep track of the development of the project, to visualize how the research plan has been affected by the data collected, and to remain aware of how he or she has been influenced by the data" (Bogdan & Biklen, 2007, p. 119). I knew from my classroom experience teaching numeracy that I would not be able to remember enough detail to record the type of field notes outlined above, and for this reason I adapted the field notes to suit my purpose (Anderson et al., 2007). My field notes were mainly written after I had watched and reviewed the video recordings (although I did write short anecdotal notes during lessons). The field notes described the problems being posed, what students were doing as they worked to solve problems, who they were working with, and paraphrased their explanations. These notes provided a comprehensive record for each day, which I was able to use when planning subsequent teaching sessions, preparing the stimulated-recall interviews, and transcribing the video recordings.

Researcher field notes provide a more personal account of the course of the research, the more subjective side of the inquiry (Bogdan & Biklen, 2007). I used my field notes to record my thoughts about various topics, such as the ethical dilemmas of being the teacher and researcher, organizational issues and possible future directions for the teaching programmes. This information was used to support data gathered by other methods during the research.

3.4.4 Documents

I used two types of document data to supplement my main sources of data collection: students' mathematics books and the group modelling books. It is important to understand the context in which the document is created and the writer's purpose, both of which I was able to verify (Anderson et al., 2007; Bogdan & Biklen, 2007). An advantage of using documents is that researchers do not have to create the data themselves (for example, by conducting interviews or observations), but are able to use data created for other purposes, and in this way are less labour intensive (Anderson et al., 2007). The documents I gathered were readily available. I was able to talk to the students if it was necessary to clarify details or seek further explanation. One limitation of using documents is that the quality of this type of materials varies (Bodgan & Biklen, 2007). This was the case with the students' mathematics books; in some just the problem answer was recorded, although most included explanations or details of workings which illustrated and provided insight into students' thinking. The group modelling books, while not created by students, provided another source of rich description of the strategies used and students' ideas (Bogdan & Biklen, 2007).

3.5 Ethical considerations

The primary ethical consideration for this study was ensuring the safety of the two case study groups. As the researcher I identified the possible risks to students as being fourfold. There were, first, issues about the disparity of status and power between students and teacher. Second, risks relating to confidentiality for each participant within the group, and, third, to confidentiality for each participant within the study, were identified. The final identified risk was a curriculum risk due to me not following the regular classroom programme.

One of the greatest challenges for researchers working with children is the disparity of power and status. Researchers need to acknowledge both this disparity and the different standpoints from which it is seen by the adult researcher and the student participants (O'Kane, 2008; Smith, 2011). Robinson and Lai (2006) observe that researchers should use their knowledge of the power relations in deciding how to inform participants (although here they are talking about other teachers). I used my prior knowledge of the students in deciding to talk to both groups and explain what I was asking them to do, and why I needed their help. I wanted to ensure they were "told as much as possible, even if some of them cannot understand the full explanation. Their age should not diminish their rights" (Fine & Sandstrom, 1988, as cited in Cohen et al., 2007, p. 54). I believe the students understood that important aspects of their experience were the objects of the research, thus ensuring their participation was authentic (Power & Smith, 2009; Smith, 2011).

It is possible that the students may have felt pressurized to volunteer, perhaps because of my position of authority or because they did not want to offend me (Bogdan & Biklen, 2007; Cohen et al., 2007). I made it clear that it would not be a problem if they chose not to participate; they would still be part of the teaching group, but would not be involved in the initial interviews and no data about them would be used in the research findings. I noted that all appeared to be enthusiastic about participating, especially the thought of having code names (one hoped he could choose his own code name and that it would be reminiscent of something from *James Bond*). Powell and Smith (2009) note that enthusiasm to participate is an indication that an attempt has been made to make the research child-friendly.

It is important to gain informed consent from students since it cannot be presumed that, because parents give consent, students are keen to participate in the research. Further, students' informed consent shows that the process of seeking permission does not involve either deception or coercion (Powell & Smith, 2009). I sought written consent from students as well as their parents. The letters specified how data would be recorded and used, particularly video data. Participants are recognizable in video data and often call each other by name (Robson, 2011). Therefore particular care has to be taken in protecting their identity. I sought permission to use the video recordings as part of my data analysis, and specified that they would not be used in the presentation of findings. Consent was received from all students and their parents.

At the time of gaining informed consent, I was aware that the devastating February earthquake had happened less than three months earlier. The post-earthquake environment meant I had a particular duty of care towards the students, because I was required to adhere to the Canterbury School Research Protocol and register details of my research (particularly the ethical procedures) with the Ministry of Education. I was aware of the effects the earthquakes were having on school-age students in Christchurch. As an insider researcher, who knew all the children, I was able to reassure them about their safety, and particularly what would happen if there was an aftershock during a lesson.

Another risk relates to confidentiality for each participant within the groups. There was a potential conflict between the need to ensure confidentiality for the students and the need to provide feedback to their mathematics teachers. I agreed to share normal classroom assessment and next step learning information with each student's mathematics teacher. No other data was to be shared with teachers. It was not possible to guarantee confidentiality within the groups as the students were all present during the teaching sessions, and the teachers knew who was in the group that came from their classroom, because it was their group.

There is also a risk related to ensuring confidentiality of the participants within the context of the study. Although it is not possible to guarantee confidentiality within the group or in regard to usual classroom assessment data, it is essential that confidentiality be preserved in the reporting of data in this study (Davidson & Tolich, 2003). I have discussed the concerns arising from the video recording of lessons above. These recordings are confidential, and cannot be shared with anyone other than my thesis supervisors. Students are identified by pseudonyms in the study. The identities of the students in the final sample group were not to be disclosed to either the teachers or to other students in the group. In this section I have discussed preserving confidentiality rather than anonymity; the participants are not anonymous because, as researcher, I know the identities of the participants and can identify a given response as coming from a particular participant (Tolich & Davidson, 1999).

The final risk was due to me not following the usual classroom programme, for example, not including a mixture of knowledge and strategy teaching each day or not following the numeracy unit plan for each group. I agreed to work with the teachers to determine a starting point for each unit, and to identify any specific needs or concerns. The structure of the numeracy programme means that groups follow group plans, thus minimizing the risk that they were not following classroom programmes.

3.6 Trustworthiness

3.6.1 Trustworthiness

In using an interpretive methodology, it is possible to replace notions of validity and reliability with considerations of the trustworthiness of the research. Guba and Lincoln (1989) identify four aspects of trustworthiness: credibility, transferability, dependability and confirmability.

Credibility is dependent on issues related to the scope of the study. These include the researcher's prolonged engagement in the field. This enables researchers to immerse themselves in the context and establish a rapport with the participants. Persistent observation contributes to credibility because it adds depth to the scope offered by prolonged engagement. Finally member checking helps to correct errors of fact and/or interpretation (Guba & Lincoln, 1989). Comprehensive and well-linked accounts that identify areas of uncertainty address credibility. Transferability is achieved when a study contains 'thick descriptions,' so that readers can apply the study to their own settings. Further, 'thick descriptions' "lend themselves to accurate explanation and interpretation of the events" (Cohen et al., 2007, p. 405). A study is dependable when the data is stable over time. The study is tracked and documented in a dependability audit of the process and method decisions taken over the duration of the research (Guba & Lincoln, 1989). It is also important to show the presence of the researcher by explicitly describing their role (Yin, 2011). The final aspect of trustworthiness is confirmability. This is concerned with assuring that "data, interpretations, and outcomes of inquiries are rooted contexts and persons apart from the evaluator and are not simply figments of the evaluator's imagination" (Guba & Lincoln, 1989, p. 243). The research must be based on an explicit set of evidence, and conclusions drawn in reference to the data (Yin, 2011). Data sources must also be tracked so that the methods and processes are explicitly described throughout the actual sequence of the research process.

In this study, my role as a participant observer and my relationship with the participants will be clearly identified, thus enabling the reader to see my position (Janesick, 2003). The conclusions I reach will be qualified by the social role I have within the research site, specifically as the teacher of the groups (Le Compte & Preissle, 1993). This will help to ensure the study is dependable.

Thick data will be collected using a variety of data collection methods, which will allow for one to be compared against another during analysis (Bogdan & Biklen, 2007). Comparing and cross-checking for consistency of information derived at different times and by different means adds credibility (Bogdan & Biklen, 2007; Guba & Lincoln, 1989; Patton, 2001; Robinson & Lai, 2006). Recording classroom observations and interviews will increase the accuracy of the data because I will not have to rely on my notes or my memory (Robinson & Lai, 2006).

It is important that studies like this one take place over time and in an authentic context. My research includes two case studies involving different groups of students. This means there will be prolonged engagement, enabling me to immerse myself in and understand the way each group uses imaging, to see how events evolve and capture the dynamics of the situations (Cohen et al., 2007; Guba & Lincoln, 1989). The prolonged engagement and persistent observation will contribute to the credibility of the data. Trustworthiness in qualitative research has to do with "description and explanation and whether or not the explanation fits the description" (Janesick, 2003, p. 69). Reporting changes to the research design helps to ensure that the explanation fits the description.

3.6.2 Changes to the research design

Changes to the sample

Originally I had planned to seek consent from three or four pre-determined students, who would have been my sample. In response to a query from the Ethics Committee, I amended my proposal and sought consent from all students in both groups. One reason for this was to avoid any student feeling excluded because they had not been asked to be part of the sample. I had planned to identify the three or four students from each group who would be the final sample soon after each teaching unit began, and to focus on them for my data collection. However, I found it almost impossible to concentrate on a limited range of students within large groups, while fulfilling my obligations as their teacher. Using the video camera enabled me to observe all students in the groups, and so the choice of the final sample was left until after each teaching unit had been completed. The advantage of this was that I could reflectively review all data to ensure my final sample represented the range of both ability and strategies for solving problems within each group.

Changes to the timing of stimulated-recall interviews

In my original proposal I planned to conduct stimulated-recall interviews after finishing the number properties phase of each learning outcome. O'Brien

(1993) claims that these interviews should be conducted as soon as possible after the recorded sessions. I found this was not possible for a number of reasons. Recordings had to be reviewed and logged before possible segments for stimulated-recall interviews could be identified. I then had to review these segments to determine whether I needed further information from students. To avoid disrupting students' classroom programmes, I collated clips from several days before asking to interview them. These interviews did not begin until the third week of group 1's unit. The interviews for group 2 were included in the final assessment interview. This was because of the time it took to review and prepare for group 2's stimulated-recall interviews, while also conducting final assessment interviews with group 1 students and planning and teaching those in group 2.

3.7 Data analysis

3.7.1 Organization of data

Video and audio recordings present researchers with a range of options for organizing data, including whether or not the whole recordings are viewed and transcribed and whether or not the researcher transcribes the recordings.

It is suggested that researchers view the entire recording initially, stopping at important or insightful events and marking them (Anderson et al., 2007). As soon as I could after each teaching session, I viewed the recordings to identify relevant sections of each video recording. I generated field notes about key events, and also copied truncated versions of important sections into an edited version of each day's lesson for ease of reference. This process enabled me to identify possible episodes of mathematical activity that could be relevant to my research questions. I used my field notes to help identify sections for full transcription. These sections were transcribed fully, using both the relevant edited and full video recordings. In the transcript, each teaching session was coded and annotated to include details of the learning outcomes and mathematics problems. I reread each transcript, correcting errors and returning to the video to clarify points or add further details. The advantage of transcribing the recordings myself was that I could review the videos as many times as I needed to ensure the transcripts were as accurate as possible. I could also use my knowledge of the context or other data sources to help decode those parts of the recording that were difficult to hear. The disadvantage of transcribing the recordings myself was the amount of time it took. The recordings for group 1 were transcribed after the in-class data collection had finished. Group 2's recordings were not transcribed until after the group 1 data had been analysed.

I transcribed the three in-class audio recordings as soon as possible after each session. Although the digital recorder was close to and facing the subjects, it was sometimes difficult to hear what they were saying because of other conversations in the classroom. Any gaps were noted in the transcript. Similarly, I transcribed the digital audio recordings of the initial and final interviews and the stimulated-recall interviews. Transcripts were checked for accuracy by comparing them with notes I had made during the interviews, and, when relevant, students' written recordings of their strategies to solve problems.

Group modelling books were annotated with information such as the lesson date, learning outcomes and sample problems. The modelling books also contained examples of students' solutions to mathematical problems, which were written in the modelling books (usually by me) while the students described their working. The modelling books were used in conjunction with video logs when preparing stimulated-recall interviews and as a reference during data analysis. Examples of students' written work were photocopied at the end of each teaching unit, dated and filed with other material pertaining to that student, as a reference during data analysis.

3.7.2 Analysis of data

> Imagine a large gymnasium in which thousands of toys are spread out on the floor. You are given the task of sorting them into piles according to a scheme which you are to develop. (Bogdan & Biklen, 2007, p. 173)

This is what it felt like as I collected all the data to be sorted and analysed. I chose to use a thematic, inductive approach, which allowed categories, themes and patterns to emerge from the data (Janesick, 2003; LeCompte & Preissle, 1993). A thematic approach is appropriate for open-ended research because the researcher is able to manage and organize the data (Mutch, 2005). Wellington (2000) suggests that one of the practical difficulties of thematic analysis is to develop categories that are relevant and meaningful, but which maintain the connection with the whole. One way of organizing data is to use the research questions in order to draw together all the relevant data for the issue of concern and preserve the coherence of the material (Cohen et al., 2007). This is a valuable strategy when faced with a large amount of data because the original research questions can be used to guide and plan the research (Wellington, 2000). I chose to use the research questions to analyse my data. This section discusses the data analysis process I used.

Data from group 1 was organized and analysed before I began to look at the group 2 data. This was for practical reasons; I felt it would be very difficult to code multiple sources of data from two groups at the same time. Working with the group 1 material first enabled me to establish categories and sub-categories, which could then be used to analyse data from the group 2 students in the final sample.

Transcribing the video and audio recordings myself gave me time to reflect on the data and allowed me to "[zoom] in on selected instances to probe at the micro-level…[and] add layers of complexity" (Otrel-Cass et al., 2010, p. 113). Once I had done this, I reread the video and interview transcripts and my notes to check the data for completeness and develop an overview of its contents (LeCompte & Preissle, 1993). I also looked for themes, phrases, students' ways of thinking, and repeated activities that stood out (Bogdan & Biklen, 2007). My first categories for analysis related to three of the four research questions I had asked:

- When do students image as they solve mathematical problems?
- What images do students create when solving mathematical problems?
- How do students use imaging when solving mathematical problems?

Initially, I coded data for all nine students in group 1. However, after reading and rereading the data for each category, I narrowed this down to the final sample of seven students.

I reread the data several times for each of these three categories, looking for significant words, phrases or concepts, "engaging in detective work, following hunches" and establishing links (LeCompte & Preissle, 1993, p. 247). I was aware of the importance of reading the data with an open mind so that patterns could emerge. This was particularly important because my 2007 small-scale investigation meant I had a hunch about what the research might say, and therefore had to be very aware of any preconceptions or bias I had (Anderson et al., 2007). When necessary I went back to the video or audio recordings (or to work samples) to confirm the accuracy of the transcript or add additional detail.

The sub-categories were flexible (Wellington, 2000). I experimented with a number of possible ways of organizing the data, including whether to use words and phrases commonly used by students or my interpretation of what they were saying, before deciding on the final sub-categories (Wellington, 2000). The sub-categories enabled the data to be broken apart in an analytically more relevant way (Coffey & Atkinson, 1996). Sub-categories can overlap, and there were a number of instances of this occurring in my data (Coffey & Atkinson, 1996). During the initial coding, data was included in more that one category if appropriate. A final decision about the best fit for overlapping data was made during the later analysis.

The initial three themes were later reorganized into five themes, to better reflect the range of the data and enable it to be presented in a more coherent manner. These also enabled discussion of points of tension with my original research question (Janesick, 2003), including a student who did not appear to image unless specifically asked to. The five themes and their sub-themes are described below:

1. When do students image? There are three main phases when students imaged during mathematics lessons. These are:
   - During teacher-led activities
   - When solving problems independently
   - When problems are difficult.
2. What images do students create during teacher-led and independent activities?
   - Creating an image of the materials used during teacher-led instructional activities to make problems

- Imaging using a transformed image. This is an image of something other than the materials used to make problems during teacher-led instructional activities.

3. Student's explanations of how they use imaging. This analyses the students' explanations of when and how they used imaging, and how and why it helped their learning:
   - To 'see' the number or the problem
   - To 'understand' the problem
   - To keep track of their steps as they solve the problem.
4. Teacher's analysis of how students use imaging to support their developing knowledge. This is divided into two sections:
   - The role imaging plays in scaffolding students' learning
   - How imaging is used over time. This looks at how students used imaging over the course of the mathematics units.
5. The times students do not use imaging to solve problems. This is divided into two sections:
   - When problems are easy. This describes what students do when they attempt to solve a problem they think is easy.
   - An example of a non-imager. Not all students in the study imaged. This sub-category looks at one student who does not appear to have used imaging to help him solve problems unless specifically asked to do so.

3.8 Summary

This chapter has described the methodology and research design of this study. Data collection methods have been described and discussed, together with the ethical considerations surrounding the study and issues relating to trustworthiness. The methods for analysing data have also been described. The final section of this chapter defines the data analysis categories. These categories are also an organizing structure for the reporting of the data analysis in Chapters 4 and 5.

Chapter 4: Results 1

This chapter reports on the analysis of data from the two groups' teaching units of addition and subtraction of decimals and whole numbers to examine the questions: When do students image? What do students image? The results pertaining to the first question are broken into three sub-categories: students image when they are supporting developing knowledge, when problems are difficult, and when they explore problems independently. The data related to the second question is reported in two sections: picture images of the materials used to model problems, and transformed images. The chapter includes some contextual information about the unit activities for each teaching group and the materials used to assist with clarity and introduce the focus students in the groups.

4.1 When do students image?

4.1.1When they are supporting developing knowledge

This section describes imaging in relation to teacher-led activities, particularly the targeted movement from the Using Materials to the Using Imaging phases of the Strategy Teaching Model (STM). Teacher questioning was designed to support the students moving to and within the Using Imaging phase and to encourage them to talk about the materials and how they could manipulate them. During these activities the groups worked together. Think-pair-share was used, students sharing ideas with their partner before reporting back to the groups.

Initially the decimals unit (following the learning activity, *Pipe music with decimals* (Ministry of Education, 2008c)) focused on developing students' knowledge of decimal place value, and the relationship between decimal and other fractions (for lesson plans, see Appendix D). Students were asked to build decimal fractions on decipipes and linked cubes and use these to discuss the relative sizes of the fractions and the meaning of the digits in the number (for illustrations of the materials used, see Appendix C). Students were also asked to image decimal fractions and describe the pieces they would thread on to the decipipes if they were building the number. As they described the numbers, they pointed to the decipipe pieces and used their hands to indicate relative sizes. For example, Ruth described how she would make 0.307. She said that she would put on “three of those that are about that big” (she indicated the size of a tenth with her hands on an empty decipipe and then showed how much space would be taken up by three-tenths), “none of the little ones” (hundredths), and “there would be seven of the silver round ones” (the thousandths).

After introducing each new strategy with materials (for example, decipipes or bundled ice cream sticks), students from both groups were asked to image the materials, either by looking at them but not touching them, or by mentally imaging what they looked like. Two activities from the first week of the decimals unit show group 1 students imaging made decipipes. In the first, they were asked to explain how they would add 0.5 and 0.37. Students pointed to the decipipes as they discussed how they would first put the tenths together and then add on the hundredths. The second asked them to image made decipipes showing 0.5 and 0.63 and to explain why these addends did not equal 0.68 (a problem designed to generate discussion about some of the common errors made when adding decimal numbers). As she looked at the decipipes, Ann talked about starting with 0.5 because it was easier to add other numbers on to it and then adding the six-tenths and the hundredths to get the correct answer of 1.13. Simon ‘saw’ what was wrong with the original problem, saying that the 5 was added to the hundredths not the tenths.

During the second week of the decimals unit, decimats were introduced as an alternative concrete material to the decipipes. Group 1 students used mental picture images to predict the steps they needed to take, and then coloured in the relevant sections as they worked to solve difference problems by reversing the subtraction equation and adding. Jill was asked what she would do first to find the difference between 0.67 and 0.8. Looking at the decimat, she said that, first, she would colour in 0.03, which would mean that 0.7 would be coloured in. Jill said it was easy to see how to solve the problem using the decimats, particularly being able to colour them in (Using Materials) to check that her predictions were correct. Simon used a combination of imaging and whole number knowledge to find the difference between 0.66 and 0.9. After using a mental picture image to predict that his first step would be to colour in 0.04 to take 0.66 to 0.7, he said he would think of whole numbers to find the difference between 7 and 9, before converting it back to a decimal.

The final strategy for group 1, rounding to a tidy number and compensating, was introduced during the third week of the unit. Having manipulated materials, group 1 students were again asked to create mental picture images of made decipipes. Ruth described rounding 1.09 to 1.1, and that adding 0.68 would then be easy. As he looked at the decipipes, Simon asked why the equation hadn't been changed to 1.1 + 0.67 (from 1.09 + 0.68), saying that we had borrowed a hundredth to make 1.1 and so had to pay it back.

Group 2 students imaged visible materials (by looking at them but not touching them) as part of the instructional activities for all three of the strategies introduced during the unit. Caroline imaged made groups of ice cream sticks as she explained how she would add 31 and 27 during the first week:

> I had the 3 separate with the 1. Then I split the 27 into 20 and a 7. I put the 20 on to the 31 so it was 51. All I was left with was the 7 and I put the 7 on the 1 so it was 58.

Tom imaged visible film canisters containing counting beans as he described how he would add 54 and 25: "In my head, plus one of the tens, gives 64, then plussed the other to 74, plus 5 which equals 79."

Hundreds boards were introduced during weeks 2 and 3 to model addition problems involving change unknown. Emily described how she imaged a visible hundreds board as she solved 55 + ? = 77:

> I added a 2 so it equaled 57. Then I just added on 20. I went 1, 2 [showing how she counted across the hundreds board] and then I just went straight down.

Students in both groups were encouraged to create mental picture images of the materials, and describe how they would manipulate them to solve problems. From group 1, Simon talked about "number shuffling" on a decipipe to solve 0.75 + 0.6 as 0.7 + 0.65. He described shuffling the five-hundredths on to the 0.6 because there were no hundredths, and so the tenths and hundredths were "free" and could be put together. Starting with a mental picture image of 0.7 on a decipipe, he added 0.3 to get to a whole, and then 0.35. Jane also used a mental picture image of decipipes as she solved 1.53 – 0.8. She used her hands to indicate the relative sizes of the pipe pieces as she described the five-tenths and three-hundredths. She said she would solve the problem by subtracting 0.8 from 1.5 and then putting the hundredths back on. Simon commented on her strategy, saying, "you put them [the spare hundredths] in the trash bag…. Then you bring them back out. You use your imagination." Ann also used a mental picture image to find the difference between 0.7 and 0.54. She explained that she was thinking about the best way to solve the problem and was thinking about the decipipes as she looked to make a tidy number (a "whole" efficiently) by rounding.

Caroline imaged shielded film canisters filled with counting beans to solve 79 – 67 during the final week of group 2's unit. She subtracted, first, the ones so she had "2 ones and 7 tens left," and then the 6 tens to arrive at the final answer of 12. Earlier in the unit, she used a mental picture image of shielded bundled ice cream sticks as she added 23 and 42:

> C: Split the 23 into 3 ones and 2 tens. I put on the 3 ones first so it was 45. Then I was left with the 2 tens so I added those on and it was 65.
>
> T: How did you add the 2 tens on?
>
> C: I know that 40, 42 plus 20 is 62, so 45 plus 20.

As the groups progressed towards Using Number Properties for each strategy, students were encouraged to image or use materials if necessary to solve problems. Jill and Ruth used number properties (and their knowledge of subtracting whole numbers) to solve 0.56 – 0.3 during the second week. Their first step was to subtract three from five (as whole numbers) before adding 0.6 back on (Figure 4.1). To check their answer was correct they decided to make the problem using materials. They made 0.56 on decipipes, and removed the pipe pieces to subtract 0.3.

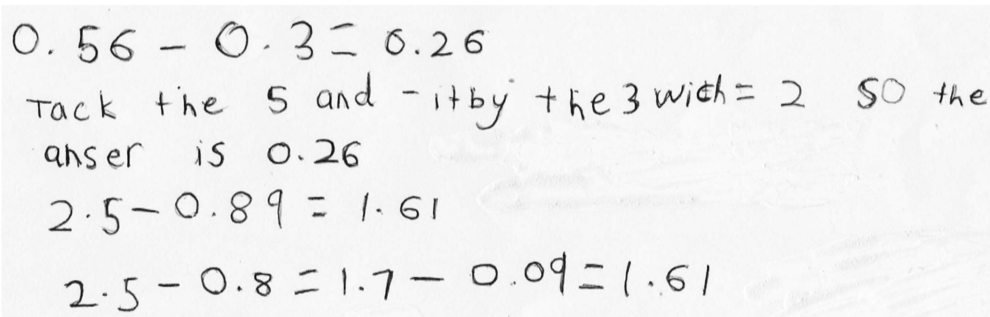

0.56 – 0.3 = 0.26
Tack the 5 and - it by the 3 with = 2 so the anser is 0.26
2.5 – 0.89 = 1.61
2.5 – 0.8 = 1.7 – 0.09 = 1.61

Figure 4.1 Jill and Ruth's solution to 0.56 – 0.3

One problem from week 3 of the decimals unit asked students to find the difference between 4.69 and 3.8. Several students chose to image. However, rather than using mental or drawn picture images of made materials, most imaged an empty number line (a transformed image), either by drawing or mentally imaging it, and described jumping forwards or backwards and then adding the jumps together. Jill described the steps she took to solve the problem using a drawn image of an empty number line:

> I started with 3.8. Then I jumped 0.2. That was 4.0. Then I jumped another 0.6. That was 4.6. Then I jumped 0.09 which was 4.69. Then I added the jumps together.

Solving the same problem, Ann drew an empty number line to help her jump back because she was not sure how to reverse and add.

Group 2 was asked to use number properties to add 265 and 134. However, Tom chose to use a transformed mental image of an empty number line: "To add the 30 I came up with a number line, and I jumped in tens – 365, 375, 385…." Emily used a mental picture image of a hundreds board as she used a combination of number properties and imaging to find the missing addend in 77 + ?= 89 (a problem the group had been asked to solve using number properties). Initially she imaged the hundreds board before recording a jump of 2 on her number line. She then added 20 on her number line without reference to the hundreds board. When asked to double check, she imaged the hundreds board again (by looking at it but not touching it) and corrected her answer.

A final example from the third week of the decimal unit shows Ruth using, first, a transformed image and, second, a picture image. Asked to solve the problem 1.6 + 0.93 by rounding and compensating, Ruth initially used a transformed drawn image of an empty number line (first jumping 0.07 and then 1.6). She stopped and looked at what she had done, before going to look at a set of decipipes on which Jill had made 1.6. Ruth looked carefully at the pipes (particularly the one with the 0.6 on it). She then returned to her drawn image. Her recording shows her thinking as she solved the last part of the problem, particularly the way she renamed one-tenth as ten-hundredths (Figure 4.2).

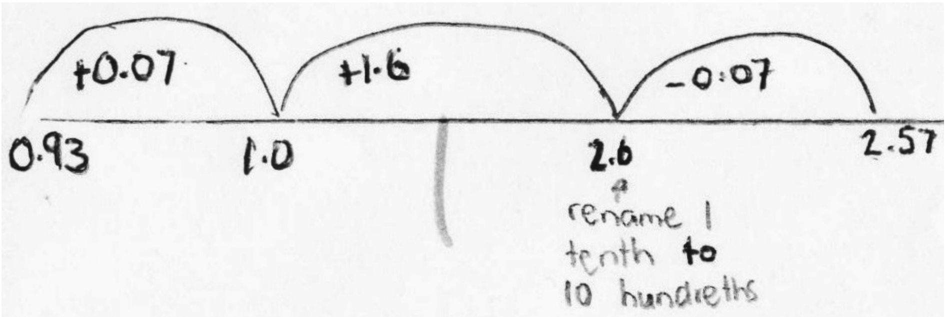


Figure 4.2: Ruth's solution to 1.6 + 0.93

Note that Ruth has drawn the last subtraction step as a hop to the right, when she should have shown the subtraction step by hopping back to the left.

4.1.2 When problems are difficult

The initial interview asked group 1 students to solve four problems. Two of the questions, questions 3 and 4, proved difficult for several students. Question 3 asked students to share 4 chocolate bars between 5 people. No one was able to solve this Using Number Properties. All were hesitant, and most attempted to use imaging to help them. Jane used a mental picture image:

> J: What you would do is lay 4 bars out. Then split them equally into fifths…. One piece from each bar to one person. Keep doing it until all the pieces are gone.
>
> T: What did you do in your head as you solved that problem?
>
> J: Imagined cubes [she explained that she was mentally imaging the linked cubes that had been used to demonstrate the strategy]. Imagined them in 5 pieces. Lay out and taking pieces. One off each bar.

Five used drawn picture images (Figure 4.3). Most drew rectangular 'chocolate bars,' which they partitioned and attempted to share equally. Using this strategy, two successfully solved the problem, and three were unable to solve it.

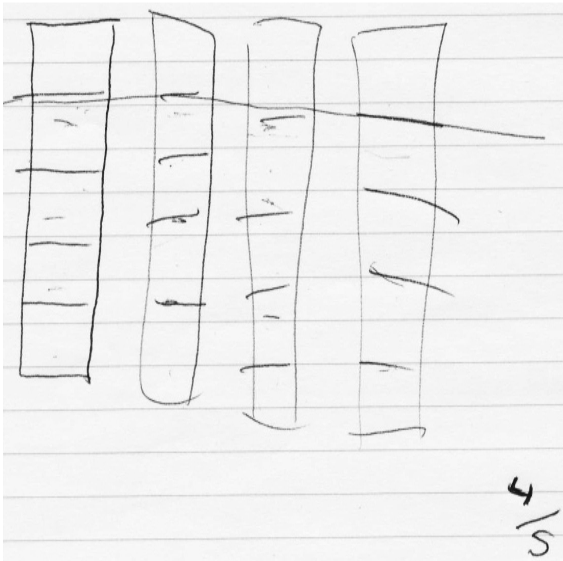


Figure 4.3 Ruth's strategy for sharing four chocolate bars between five people

Problem 4 asked how many candles would be on a whole birthday cake if there were 12 candles on two-fifths of the cake? Four students decided to use a drawn picture image of the problem. All drew a circle, which they divided into five pieces. Two also drew the '12 candles' (actually dots) shared between two pieces of cake (Figure 4.4).

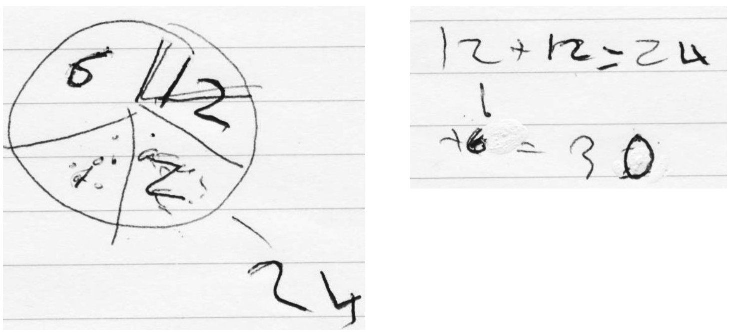


Figure 4.4: Solving 2/5 of ? = 12, using repeated addition

All solved the problem correctly, although only two used the taught strategy of using multiplication and division. Most of the remaining students used repeated addition. For example, Simon said:

> two-fifths, that's 12, then you eat another 2, that another 12, so 12 + 12 is 24. But then you've just got 1 piece remaining so what you've got to do is halve the 12…so half of 12 is 6. So you go 24 + 6 is 30.

Jill was asked why she had chosen to draw the birthday cake and candles straight away. She said it was because that was the way she had learnt to work it out, and it was the only way she knew how to solve the problem.

The following vignettes illustrate some of the strategies used by group 1 when faced with difficult problems during the decimals unit. In the first, Simon is working to find the difference between 2.5 and 0.89, which students were asked to use imaging to solve (although materials were available both to make problems and to image) and place value decomposition. The strategy had been demonstrated and practised using materials and then imaging (problems were made on decipipes, which students looked at but did not touch). Simon attempted to solve the problem using number properties, subtracting first 0.5 and then 0.3 to get to 1.7, before adding the hundredths to get an answer of 1.79 (Figure 4.5). He was not sure why his answer was wrong, so I suggested he made the problem using decipipes to check where he had made a mistake. Simon subtracted 0.8 by manipulating the materials. He then imaged to find the answer – he looked at the 1.7 on the pipes and at a box containing the hundredths and said, "Ah, now I've got it…." He explained that imaging the pipes had helped him realize his mistake, and that he had to subtract nine-hundredths. He now knew how to do this because he knew that one-tenth was the same as ten-hundredths.

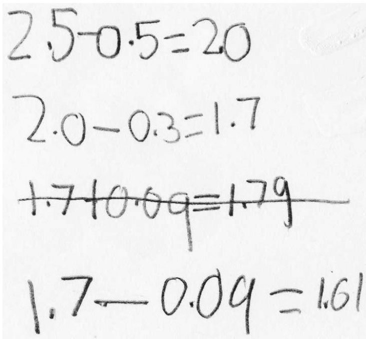


Figure 4.5: Simon's recording as he used a combination of number properties, materials and imaging to find the difference between 2.5 and 0.89

Many members of the group used transformed images of empty number lines to support their thinking when problems were difficult. Ann drew an empty number line rather than using number properties to solve a problem asking her to find the total weight of a cat and dog before they were put on a plane (3.457 + 7.543). Initially, she thought the problem was very hard and would take ages to solve. Her strategy was to keep 7.543 as a number and then add the 3.457 (using place value decomposition). However, using her knowledge of compatible numbers, she realized that she could add the 0.007 and 0.003 to make 7.55, which she recorded at the start of her number line. She says that, "Once I had changed it I could see that I could do it in one jump," and she was able to solve the problem.

David used a different strategy when problems were difficult. Asked to find the difference between 0.54 and 0.7, David initially made 0.54 on a decipipe. He then drew picture images representing 0.7 and 0.54 (Figure 4.6). He added 0.06 by drawing the hundredths on to the image representing 0.54. Finally, he drew a transformed image of an empty number line to solve the rest of the problem. David also drew picture images of decipipes during the final assessment (see Figure 5.1). Using these he was able to successfully solve the first two problems.

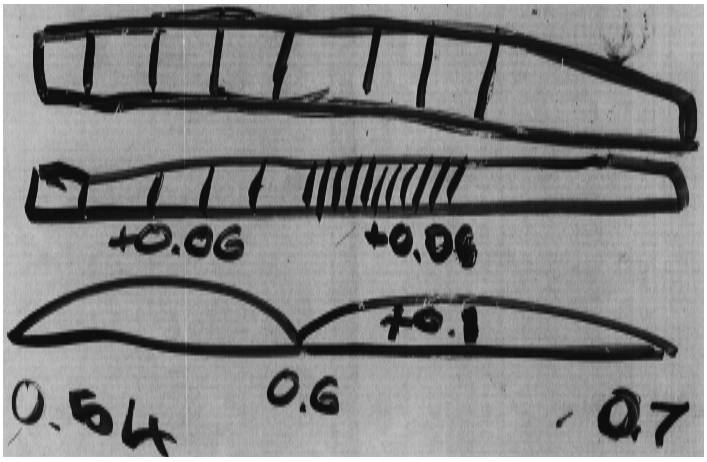


Figure 4.6 David's strategy for finding the difference between 0.54 and 0.7

The final activity in the decimal unit was based on *Create a question* (nzmaths, problem solving, level 4). This asked students to use their knowledge of decimal numbers to create problems for others to answer. Each group was given an answer and asked to write three problems to match the answer (using the three strategies taught during the unit). No one attempted to use only number properties as they wrote their problems. All decided to use a transformed drawn image. Jane explained her reasons for using a transformed drawn image: she had to "think in different ways…to figure a question that had that answer," rather than just answering a question. Having chosen a "random number" (as Jill described it), all groups used a transformed image of an empty number line to 'see' what they were doing, particularly to see the numbers in a jump and to work out how they could get to their answer. Three of the four groups were able to successfully devise problems using at least two strategies, and one group wrote story problems for each equation.

The final unit assessments provide further examples of the strategies students use when problems were difficult. At times students used imaging rather than number properties. Jane created a transformed mental image of a Venn diagram to find the difference between 6.13 and 5.8, and Jill drew an empty number line to help her see and remember what she had done as she found the difference between 6.48 and 3.92. On other occasions I suggested that students used imaging because they had attempted to use number properties, and were either unable to solve the problem or had solved it incorrectly. In discussing how students tackled difficult problems in the final assessments, it is interesting to note that several students reverted to favourite strategies (not necessarily the strategies that had been taught or strategies that were the most effective way of solving the problem). For example, Peter (see section 5.3.2) used total place value decomposition (although he had said that his favourite strategy was rounding). Ann jumped up and back along a transformed drawn image of an empty number line to add and subtract. Caroline looked at the digits (making no reference to place value) and used total place value decomposition to solve 783 – 151: "I went 7 – 1 equals 6, 8 – 5 is 3, 3 – 1 is 2." It is interesting to note that in her other explanations during this assessment she had used place value when describing numbers.

4.1.3 Imaging when students explored problems independently

Four activities, three spread over the three weeks of the decimal unit and one from the addition and subtraction unit, are included in this section. Students worked independently, generally with a partner, to explore the problems included in these activities. Materials were available for students to make the problems or to support imaging if needed. Two of the activities used resources from the nzmaths website problem solving section (*Create a question* and *Reversing numbers*). A third, Make 0.5 and Make 2.07, was adapted from the nzmaths problem solving activity, *Make 4.253*. All three were open-ended activities asking students to come up with a variety of responses. The fourth activity asked group 1 students to choose appropriately from the range of strategies taught during the unit to solve problems. Students' approaches to the *Create a question* activity have been discussed in section 4.1.2.

Make 0.5 (and an extension activity of Make 2.07) was included to support group 1 students' developing knowledge of decimal place value (particularly the relationship between tenths and hundredths) during the early stages of the unit. Decipipes were available for students to make equations and they were asked to justify their choices by recording their ideas on an empty number line. Working together, Ruth and Ann had successfully found a number of combinations using tenths. Challenged to use tenths and hundredths, Ann recorded the equation 0.32 + 0.18 = 0.5 in her book. She was not sure whether or not she was correct, so I suggested that she checked by making the problem on the decipipes. Ann used a combination of Using Materials and Using Imaging to check her equation; after threading four-tenths on to the pipes she said "Oh yes it will work…because 10 of these [hundredths] make one-tenth." She explained that seeing the "tenths and stuff" had helped her to realize that her equation was correct. As she fed the ten-hundredths on to her decipipe, she commented that there were actually "heaps" of ways of making 0.5, "millions and millions."

Towards the end of the decimal unit, students were asked to use number properties or, if necessary, imaging to solve three problems that practised the strategies taught during the unit. Some, including Jane, worked quickly through the problems, using number properties and recording just the answer (section 5.3.1 for a discussion of her solution to a problem asking students to add 1.6 and 0.93). Others found the problems tricky. Ruth and Jill used imaging (transformed drawn images of empty number lines recorded in their books) to solve all three. In the example described here, their use of imaging not only enabled them to solve a tricky problem, and to justify their solution, but to develop a strategy that had not been included in the unit, and which was a more efficient way of solving the problem. Problem 2 asked students to find the difference between 1.68 and 3.54. Ruth and Jill chose to reverse the subtraction problem and add (Ruth reminding Jill that they were not adding the two numbers together, but were finding the missing addend). Initially they attempted to solve the problem by adding in little jumps from 1.68 (first to 1.7), and recording these on a number line. However, they were unable to solve the problem. They then decided to add 2 to 1.68, although they knew this jump would take them beyond the target number and they would have to subtract something to get to the correct answer (Figure 4.7).

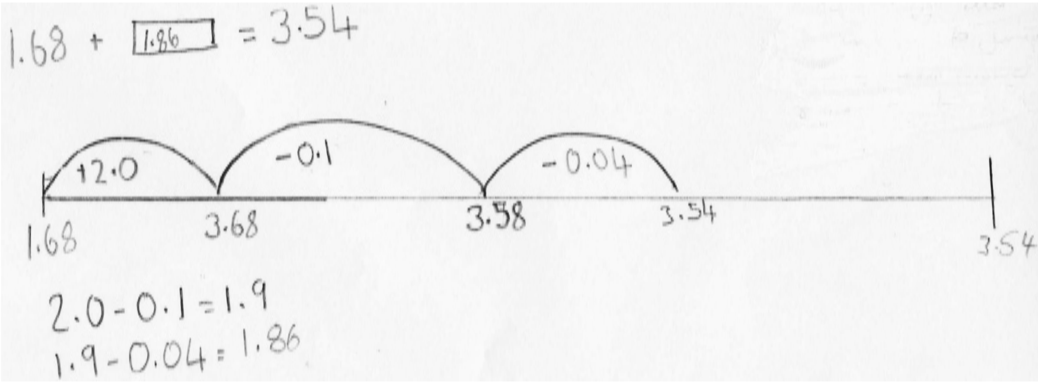


Figure 4.7: Ruth and Jill's strategy for finding the difference between 1.68 and 3.54

Still questioning whether or not this was an appropriate strategy, they subtracted 0.1 and then 0.04 on their empty number line to reach 3.54 and the correct answer of 1.86. Note again that, in their representation, Ruth and Jill have shown the subtraction part incorrectly by continuing to move to the right as they subtracted. The following exchange then took place between them:

> R: Mrs C will want to know why.
> J: Just tell her the truth. She'll know why we did it.

The "truth," as they explained in a later interview, was that their way was easier than the strategies I had suggested. Unable to get to the right answer by jumping along the empty number line in small steps, they decided to try one big jump and then subtract. Jill explained that the numbers in the equation (0.68 was greater than 0.54) made their initial strategy difficult, because it was difficult to add all the jumps together, and because, if they "jumped to make it

3 point something it would always be higher than we wanted it to be. So we jumped 2.0 anyway and then we minused it back…." Ruth added, "It was easier to jump big whole numbers than going up in tenths."

Reversing numbers was included in the addition and subtraction unit to enable students to practise addition of two-digit numbers and to give them an opportunity to 'play' with numbers and look for patterns. Materials were available if students needed to make the problem, or to support imaging, and students were asked to show their working. Emily manipulated the numbers to find the answers. As she added 34 and 43, she put her fingers on the pairs of numbers representing the tens and ones and said "seven, seven," before recording her answer, 77. Her partner, Caroline, chose to use materials (animal strips) to make 52 + 25, recording her steps on an empty number line as she moved first the 5 and then the 2 tens to find the answer. Tom used a transformed drawn image of an empty number line to solve all the problems he devised. He said his first step was to jump 40 as he added 54 and 45 (Figure 4.8). Solving another problem (53 + 35) he added 30 to 53 by jumping 20 and then 10. He then checked that he had added the ones correctly by using his fingers (materials).

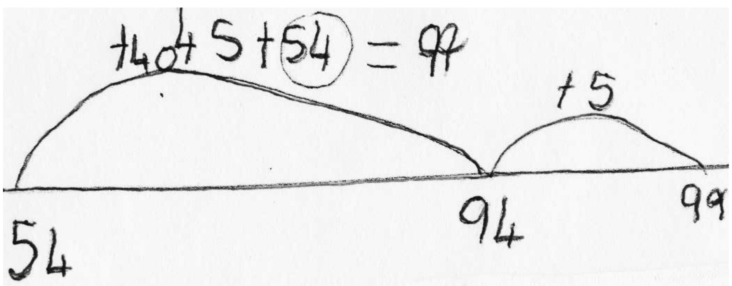


Figure 4.8: Tom's empty number line showing his solution to 54 + 45

4.2 The types of images used by students

The images used by students fall into two categories: picture images of the materials used to make the problem (4.2.1), and transformed images (using their own image to represent the problem) (4.2.2). There are also examples of students using a combination of images to solve problems (4.2.3).

4.2.1 Picture images of the materials used to make and solve problems

The picture images of the materials used to make and solve problems have been broken into two sub-categories: mental picture images and drawn picture images.

Mental picture images

Students used mental picture images of the materials used to represent the problems during both teacher-led and assessment activities. These images were created by students describing how they would manipulate imagined or shielded materials, or by students describing how they would manipulate materials they could see but not touch.

During the initial interviews, two group 1 students attempted to use mental picture images of either the numeracy linked cubes or drawn rectangles to solve $4 \div 5$. Jane successfully solved the problem by mentally imaging linked cubes and taking off cubes (pieces) to give to each person. Although Ann did not solve the problem correctly (she couldn't give the final answer of four-fifths), she described giving the five people one-fifth of each bar. Tom and Emily used mental picture images (of a hand) during group 2's initial interviews. Tom talked about counting up on fingers in his head as he found compatible numbers that added to 10 to answer $3 + 8 + 6 + 7 + 4 + 2$. Emily described imaging a big hand to help her count backwards to solve $53 – 5$. During teacher-led activities, group 1 students were asked to use mental picture images to describe what decimal numbers would look like if made on decipipes. Early in the unit, descriptions of the imaged materials were often closely linked to the physical appearance of the materials. Explaining what

0.307 looked like, Ann said there would be “three of those [pointing to the tenths], none of those [pointing to the hundredths] and then you’ve got seven of those metal… [the thousandths].” This is in contrast to the way decimal numbers made on decipipes were described by students later in the unit. In week 2, Peter described what 0.413 would look like if made on decipipes. He made no reference to the physical appearance of the decipipe pieces that represented each decimal fraction, instead talking about the number of tenths, hundredths and thousandths. Ann used a mental picture image (which she described as creating pictures in her head) of tenths and hundredths to help her find the best way to work out the difference between 0.7 and 0.54. She said the mental picture image helped her to solve the problem by rounding to a tidy number. David’s descriptions of his mental picture images continued to have a close link to the physical appearance of the materials throughout the unit. Asked to describe what 1.8 would look like if made on a decipipe, he said it would be “one whole and another one [decipipe] with eight-tenths.” David solved the rest of the problem (1.8 + 0.4) by mentally imaging a diagram of decipipes, which I had drawn. Asked what he would do next, he replied, “Get that [pointing to 0.8] to a round number,” which he did, before adding the final 0.2.

When I specifically asked group 2 students what a mental picture image of their favourite materials would look like, their descriptions were closely related to the physical appearance of the materials. An example of this is Emily’s description of 45 as, “4 jellybean packets and 5 single jellybeans by itself.” However, generally, in their descriptions of their mental picture images, group 2 students did not talk about the physical appearance of the materials. Emily said she “saw the notes behind the screen” as she created a mental picture image of numeracy money (268 had been made using money and then shielded) to solve 268 – 145. Although she said she saw the notes, she talked about hundreds, tens and ones as she described how she would solve the problem. Emily also used a transformed drawn image of an empty number line as she solved this problem. Asked how she had subtracted 40 from 168, she said, "I thought in my head, I know that 6 minus 4 equals 20, I just knew that 40 plus 20 equals 60, and then I know it backwards."

Students also mentally imaged decipipes, decimats, and the addition and subtraction materials by looking at them and describing how they would manipulate them if they could touch them. During the first week, group 1 imaged made decipipes as they discussed solutions to addition problems designed to highlight some common decimal place value errors and confusions. Ruth pointed to made decipipes and indicated the size of the tenths with her hands as she explained how she would add 0.5 to 0.8. Ann also pointed to made decipipes as she explained why 0.5 + 0.8 did not equal 0.13, commenting that the numbers were tenths and not hundredths. In week 2, Jill imaged made decipipes as she described her strategy for finding the difference between 2.5 and 0.89. Having subtracted the 0.8 to get to 1.7, Jill said that the next step could be to take off one-tenth and add a hundredth (see also section 4.1.1).

Tom, in group 2, imaged a visible hundreds board as he solved 1 + ? = 23. He described first making two jumps across to get to the 3 and then jumping down 2 tens to get to 23. Caroline imaged visible counters (in bags of 10 and separate ones) as she added 46 and 33. She said she would move the 3 off the 33 and add it to 46 and then add the 30 in tens, 49, 59, 69, 79.

Drawn picture images

During group 1's initial interviews, students drew diagrams of the materials that had been used during teacher-led activities to introduce and practise the strategies. Three students drew rectangles to represent chocolate bars as they attempted to solve the problem, 4 ÷ 5. As they divided the rectangles, they talked about cutting the bars, counting the pieces, and giving pieces to each person, thus reflecting the context of the problem. One student also drew five stick figures to represent the people (Figure 4.9). His diagram showed how he would share one chocolate bar between the five people. Having shared one bar, he successfully solved the problem, saying that as there were four chocolate bars, each person would get four-fifths.

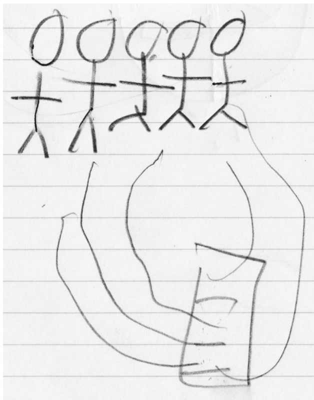


Figure 4.9: Simon's image of how he would share one 'chocolate bar' between five people

Four students drew circles to represent a cake when asked how many candles would be on a whole cake if there were 12 candles on two-fifths. Two also added dots to represent the 12 candles (counters had been used in the original teacher-led activities) (see Figure 4.4). Again, the students referred to the context as they talked about their solutions, describing candles, splitting the cake, eating, and the number of pieces.

One group 1 student, David, frequently used drawn picture images during both teacher-led and independent activities. He often chose to create an image of the problem by drawing and labelling representations of the decipipes. Figure 4.10 shows how David used his drawing of decipipes, together with a mental picture image, to help him find the difference between 1.6 and 0.59. His written recording shows his steps.

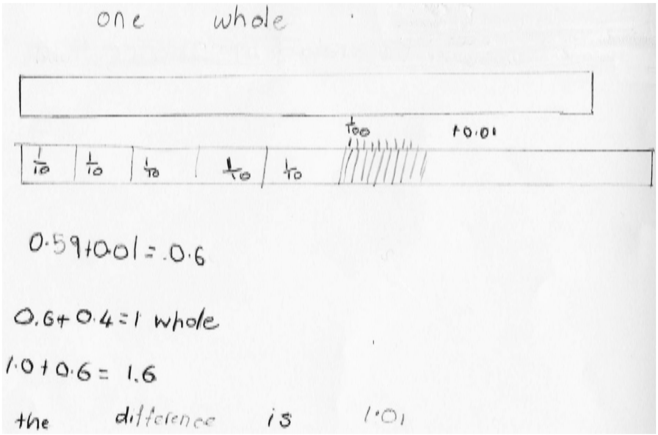


Figure 4.10: David's drawn picture image and recording of his steps to find the difference between 1.6 and 0.59

4.2.2 Transformed images

Students also used drawn and mental transformed images of something other than the materials used to represent and solve problems during teacher-led instructional activities. Often this was a generic image that could be used to solve a wide range of problems.

Jane (group 1) sometimes used a transformed mental image if a problem proved difficult to solve using number properties. She described imaging "those two circles that are kind of like…Inquiry [a Venn diagram] and it works." She said she used the common part of the Venn as a place to store numbers, and manipulated the numbers in the outer circles (although the exact use for each area depended on the problem). Figure 4.11 shows what her transformed mental image looked like as she found the difference between 5.8 and 6.13; here she has used the central part of the Venn to store each step of her solution.

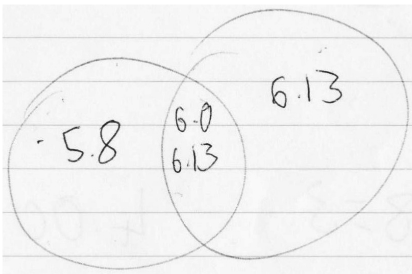


Figure 4.11: Jane's transformed image of a Venn diagram

The transformed image most often used was that of a drawn (transformed drawn image) or imagined (transformed mental image) empty number line. In their explanations of how they solved problems, students often described how they jumped up or back along their drawn or mental images of empty number lines. In considering the examples to be included in the section, I have used students' descriptions of their thinking as an indicator that they were imaging rather than just recording their working.

During the first week of the decimals unit, Simon explained that his solution to 0.7 – 0.54 was "in my head," and described jumping back along a number line in tenths and hundredths to solve the problem. Ruth and Jill also solved this problem by mentally imaging a number line, although they both chose to reverse the problem and add (first 0.06 and then 0.1). The next two examples come from weeks 2 and 3 of the addition and subtraction unit. Tom said he "came up with a number line and…jumped in hundreds and tens" as he added 265 and 134. He used the same strategy to find the difference between 64 and 31, saying that he "put a number line [in his head] and jumped back in tens" to subtract the 30, which he then recorded on a number line as a single jump of 30.

There are many instances throughout the decimals unit of students creating drawn images of their strategies by using empty number lines. They described jumping along an empty number line, jumping too far and then jumping back, big and little jumps, using the empty number line to see how far they still had to jump or to see the problem, or using the empty number line to help them to solve problems step by step.

Group 2 students also used empty number lines to create transformed drawn images of their strategies. At times they described jumping up and back as they solved problems. On other occasions they talked about "minusing," "taking off," or "counting up." Emily's explanation of how she solved 783 – 151 in the final assessment is an example of this. Having said that she needed to draw an empty number line to help her solve the problem, she described "minusing" first 1 and then groups of 10, counting her jumps backwards to make sure she had subtracted 50, and then counting back in tens as she labelled her number line ("772, 762, 752, 742, 741"). Emily also used a transformed drawn image of an empty number line to find the missing addend in 47 + ? = 123 (an extra for experts question on the final day of the unit) (Figure 4.12).

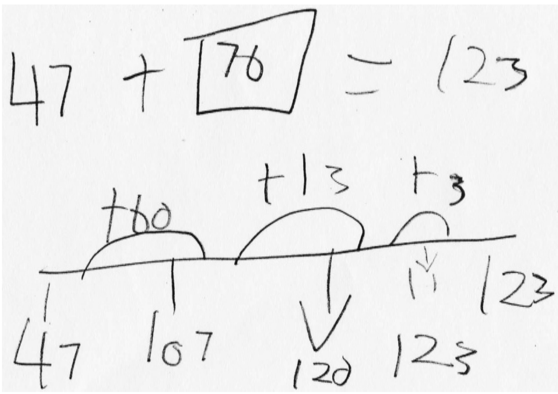


Figure 4.12: Emily's strategy for solving 47 + ? = 123

Her description of how she solved the problem shows she used derived facts as part of her strategy: "40 plus 60 equals 107. Then I added 13. Cos I can add to the 20 quickly. I know that 3 + 7 is 10 and add another 10 it is 20."

Students' reasons for using transformed drawn images of empty number lines will be discussed in the section 5.1.

4.2.3 Using a combination of images

At times students used more than one kind of image as they attempted to solve a problem. Simon attempted to use a mental picture image to solve 4 ÷ 5 in the initial assessment for group 1. He talked about "seeing" if he could divide the chocolate bars into halves or quarters, but that it was not working. It was suggested he tried to use a drawn picture image, which he did, drawing a diagrammatic representation of the problem and solving it successfully (see Figure 4.9). David used different categories of image as he worked to find the difference between 0.67 and 0.8 by reversing and adding (a problem he found tricky because he had to reverse and add to find the difference). Initially he was asked if he could use a mental picture image of a made decipipe (representing 0.67) as a starting point towards solving the problem. David immediately drew picture images of both 0.67 and 0.8 on a whiteboard. Looking at both the made decipipe of 0.67 and his drawing, he added three-hundredths to make another tenth. However, he was unable to solve the problem by Using Imaging, so it was suggested that he tried to solve it Using Materials, which he did successfully. David described the drawn picture image as a draft and said that he had needed to use materials and add and subtract the pipe pieces to solve the problem.

This chapter has reported the results in relation to the questions, when do students image? And what do students image? Students used imaging in response to teacher-led questioning and activities, as they developed their knowledge about the concepts and strategies taught, when problems were difficult and they were unable to solve them using number properties, and when they were able to explore problems independently. Students used both drawn and mental picture images of the materials used to make problems, and drawn and mental transformed images. At times they also used a combination of images as they attempted to solve problems.

Chapter 5: Results 2

This chapter focuses on why students chose particular mathematical resources as they attempted to solve problems. Section 5.1 is drawn from discussions with students about how they solved problems. Section 5.2 describes the role of imaging in supporting students' learning, including how imaging is used as a scaffold and how imaging is used over time. Finally, section 5.3 identifies times when students don't image. This section reports on, first, students' strategies when problems are easy, and, finally, describes the actions of Peter, a student in group 1, who appears to be a 'non-imager.'

In their descriptions of the strategies they used for solving problems (and their reasons for choosing a particular strategy) students indirectly describe some of the phases of the Strategy Teaching Model (STM). By this I mean, that they do not use the names of the phases, but instead might comment that they had just used numbers, or that they were thinking in their head. They also described the types of image they had chosen (for example, how the diagram they had drawn helped them), and, if appropriate, the materials they had used.

5.1 Students' comments about how they use imaging

There was considerable variation in the way students, especially those in group 1, approached problems and how they moved through the three phases of the Strategy Teaching Model (STM). Some moved quickly to Using Number Properties (see sections 5.2 and 5.3), while others moved backwards and forwards between and within the phases of the STM. As they worked to solve a problem, students appear to have selected the phase, whether materials, number properties or an imaging phase, most appropriate to their needs. If a problem proved more difficult than anticipated, some students moved independently to use another phase; others were supported to do this by teacher intervention or questioning. The boundaries between phases were often blurred – for example, students might start to solve a problem using number properties, then switch to materials to solve another section of the problem, before finally using a mental picture image to solve the final part of the problem (see also sections 4.1.1 and 4.1.2).

Some of the students' reasons for choosing the particular phases they did are discussed in this section. In reporting these results I have looked at what the students said about the way they used imaging. Much of the description uses students' own words, both as they solved problems and shared solutions and from later stimulated-recall interviews. Two words, in particular, were often used by students: 'see' and 'understand.' I have chosen to retain these words, as they represent the students' voices. I have placed both within inverted commas to indicate that they were the actual words used by the students.

5.1.1 To 'see' the number or the problem

Several group 1 students used drawn picture images of the fraction problems during the initial interviews. Ruth drew a diagram to solve the problem, 2/5 of ? = 12 (see Figure 4.4). She commented that the drawing helped her to 'see' that there was "one bit that's uneven and the rest of them are 12." Decipipes and decimats enabled many members of group 1 to 'see' the decimal numbers and how they could be added and subtracted. Here they are talking about creating mental picture images (by looking at materials but not touching them) and manipulating them (Using Materials). Ann used a combination of Using Materials and Using Imaging to check that 0.32 and 0.18 equalled 0.5. She commented that the decipipes helped her to 'see' the "tenths and stuff" because she could 'see' the relative sizes of the numbers. Imaging and manipulating the decipipes showed her the meaning of the numbers; how the tenths and hundredths related to each other – how they fitted together.

Simon used a combination of Using Materials and Using Imaging to find the difference between 1.6 and 0.81. Making 0.81 on the decipipes and then using a mental picture image of the pipes showed him how much he had to add on, and how he could use rounding to solve the problem. Working with Simon, Ann also used a combination of Using Materials and Using Imaging. She said that the decipipes showed her how the number could be cut into tenths and hundredths, and what each part meant. David often used drawn picture images as he solved problems. He said drawing the decipipes helped because he could actually make the number. During the final assessment he drew images of decipipes to help him solve 3.21 + 1.96 (Figure 5.1).

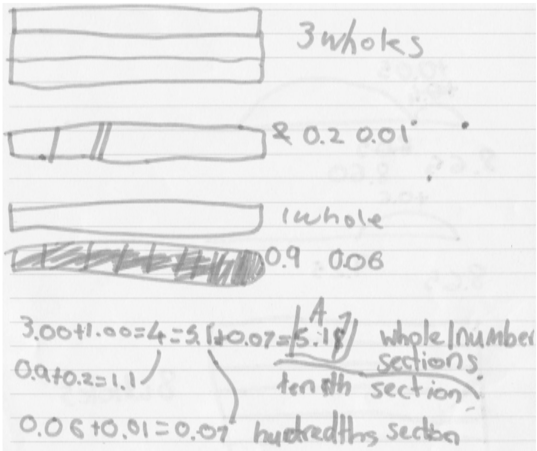


Figure 5.1: David used drawn picture images of decipipes to help him add 3.21 and 1.96

He explained that dividing the problem into the two numbers and then the "sections" in each number had given him a "clear picture" of what the problem was asking him to do. He solved the problem using place value decomposition, adding the "whole number section," the "tenths section," and the "hundredths section" separately.

In group 2, Emily described being able to 'see' (by creating a mental picture image) the shielded numeracy money as she subtracted 145 from 268. She partitioned 145 into 100, 40 and 5 and subtracted each number separately, recording her steps using a transformed drawn image of an empty number line. She described the strategy she used to subtract 40: “I know that 6 minus 4 equals 20, I knew that 40 plus 20 equals 60, and then I know it backwards.” Tom used a mental picture image of a hundreds board to help him ‘see’ problems. He gave an example of how this helped him: if he had a problem like 18 plus something equals 36, the hundreds board would provide a structure to show him how he could count down.

The group 1 students who generally created drawn images (both transformed and picture) talked about why they preferred these to mental picture or transformed images. Jill drew an empty number line to help her solve 26 + ? = 82 during the initial interview. The drawn image helped her to ‘see’ the distance between the numbers rather than having to “imagine it in my mind.” Simon said he did not really have to draw a diagram to represent 2/5 of ? = 12; he could have imaged a picture in his head. However, he said that it was easier and probably quicker to draw it. David drew the decipipes to help him find the difference between 0.81 and 1.6. He said that he preferred to have a drawn picture image because “when you’ve got it in your head it sometimes changes…but when it’s right in front of you, you can always see what’s going on.” Ann commented that having an “image of doing it, not just thinking in my head,” helped because “thinking in your head was not always right.” Imaging decimats (the decimats were in front of her) to solve 0.66 + ? = 0.9, she said the decimats helped her “to see what [she was] doing rather than figure it out in [her] head when there is a lot of stuff happening.” Simon made a similar comment about using a transformed drawn image of an empty number line, saying that if “you use a number line you can go and see it again, if it’s in your head, you can’t.” Although Tom (group 2) often used mental images (both transformed and picture), he sometimes used transformed drawn images of empty number lines. He said the empty number line he drew to help him solve 53 – 5 in the initial interview helped because "sometimes in your head you are thinking about something else (like the invite to a friend's birthday) and you lose the number."

5.1.2 To 'understand' the problem

Ann described how Using Materials and Using Imaging (particularly mental picture images) had supported her developing knowledge during the decimals unit:

> I used to get muddled up with hundredths and tenths when I started working with decimals. If I had a problem like 1.6 + 0.74 I would think the six-tenths was hundredths and think I could add the four-hundredths because I was trying to make a whole tenth. Seeing the number and making it helps me to see how it fits together…. I now understand the numbers.

Simon used a combination of materials and a mental picture image to correct the error he had made when attempting to find the difference between 2.5 and 0.89 (see also section 4.1.2). Making the problem and then imaging the made decipipes meant he could see that he had subtracted 0.8 and that he still had to subtract 0.09. The mental picture image of the pipes helped him realize that one-tenth was the same as ten-hundredths, and meant he was able to correct his mistake and solve the problem.

Creating mental picture images of materials helped Tom (group 2) 'understand' how the numbers were made up, because he could see how many bundles of sticks or containers of counting beans were used in each number. It also meant he could 'see' how to solve a problem, particularly whether he needed to add or subtract.

5.1.3 To keep track of the steps as they solve problems

Students described using mainly drawn images (both transformed and picture) as a way of keeping track of what they were doing. As the decimals unit progressed, many group 1 students moved away from creating picture images of, or manipulating, materials; instead creating transformed drawn images of empty number lines. These enabled students to go back and check to make sure they were correct, something they could not do with mental images. Jill said drawing an empty number line made it easier, and meant that she did not forget the jumps she had made at the start. It helped her to 'see' what she had done, and what she still had to do to solve the problem. It also enabled her to focus on the computation needed to solve the next part of the problem. To illustrate this, she gave an example of how she would use basic facts or her knowledge of rounding. Simon said that using a transformed drawn image of an empty number line was easier and that it "saved…doing it in [his] head." He qualified this by saying that "obviously if it's 1 + 1 I don't need to [use an empty number line]."

Jane was one of the few students from group 1 who mostly used transformed mental images. She described her mental image of a Venn diagram as being like a cabinet, somewhere that she could store numbers away until she needed to get them out. She added that it was versatile and could be used in lots of different ways (see Figure 4.11). Tom, from group 2, often used transformed mental images of empty number lines. He said that if he did not use a mental image of an empty number line (or, if he needed to, a drawn one) it was easy to forget what he needed to do. Emily (group 2) described how a mental picture image of a hundreds board helped her to keep track of what she was doing as she worked to solve a change unknown problem: "you can…go across and if you want to go down, you can go down, and then you if you want to find the answer…you just add [up the steps]."

5.2 The use of imaging to support students' learning

5.2.1 To provide a scaffold for their learning

There were a number of examples of students using imaging as they worked to solve more difficult problems, or when a problem proved more difficult than originally anticipated (see also sections 4.1.2 and 4.1.3). Simon and Jane used imaging (drawn and mental picture images respectively) to solve 4 ÷ 5 during group 1's initial interviews. Simon initially attempted to use a mental picture image. When this was unsuccessful, he used a drawn picture image of a 'chocolate bar.' Simon said this helped him to 'see' how he could divide the four chocolate bars between five people, as well as to make sense of what the problem was asking him to do. Both students said they would not have been able to solve the problem if they had not been able to use imaging.

David used a drawn picture image of a birthday cake (2/5 of ? = 12) to help him solve the problem successfully. He said that if he did not have the picture he would have been confused, and that the picture helped him divide the numbers and to put them in sections. Once he had created the drawn picture image, David solved the problem easily, using repeated addition: "I think it would be 30. Twelve on two-fifths, so 6 on each, four-fifths equals 24, so add 6 for the last piece."

Emily used a transformed drawn image of an empty number line to help her solve question 4 (523 + 246) during the final assessment (Figure 5.2). She started to solve the problem using number properties, adding 200 to 523, but drew an empty number line to help her add the tens and ones, saying that the numbers at the end were bigger and that made it harder to add. Once she had done this, Emily used derived facts to solve the problem: "I know that 2 plus 4 is equal to 6. And then I know it in my tens…. Three plus 6 is 9 and I know that 6 plus 4 is 10 so I just take away…1."

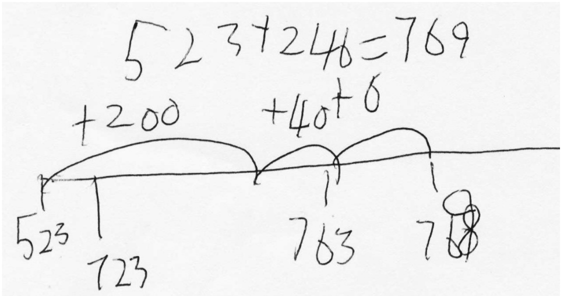


Figure 5.2: Emily's transformed drawn image of an empty number line

On many occasions, group 1 students went straight to the Using Imaging phase of the STM to provide a scaffold as they attempted to solve problems they found difficult. David used drawn images to help him successfully solve difficult problems. He often chose to use drawn picture images of decipipes, although towards the end of the unit, he was using transformed drawn images of empty number lines. He said an empty number line helped him solve the problem step by step, and he could see the steps he had taken.

During the final week of the unit, group 1 students were asked to find the difference between 3.8 and 4.69. Although it was intended that students used number properties to solve this problem, materials were available and they were encouraged to image if necessary. Jill and Ann used transformed drawn images of empty number lines (see also section 4.1.1), while David chose to use a drawn picture image. He drew and labelled decipipes to represent the two numbers. As he imaged the drawing, he recorded the steps he took on both an empty number line and in a series of equations.

Ruth and Jill used drawn transformed images of empty number lines as they worked to find the difference between 1.68 and 3.54 (see section 4.1.3). As well as representing the problem, they used the drawn empty number line to support discussions about how to solve the problem, to try different possibilities, to correct errors, and to build on each other's ideas. They were able to work out (and then explain and justify) an effective strategy that had not been taught during the unit. Jill used a drawn transformed image of an empty number line and a similar strategy to help her find the difference between 6.48 and 3.92 in the final assessment (Figure 5.3), although she made an error in the final calculation. After adding 0.06 to bring 3.92 up to 3.98, she added 3. Jill said she had "got higher than what I was meant to have," and that she will have to subtract 0.5 to get back to 6.48. She said that the empty number line helped her to remember what she had done, because sometimes she forgot that she had made a jump. Jill has again represented the subtraction jump incorrectly by continuing to move to the right, rather than jumping back.

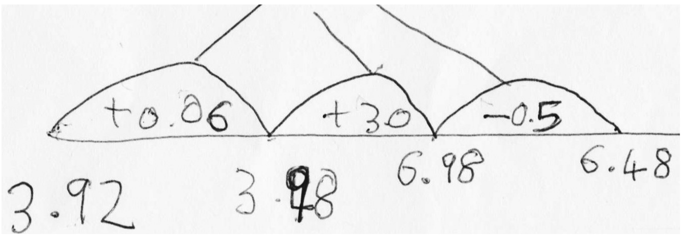


Figure 5.3: Jill used an empty number line and tidy numbers to jump beyond the answer, before subtracting

All group 1 pairs used transformed drawn images of empty number lines as they worked on the *Create a question* activity (see also section 4.1.2). Jane commented that the activity was "a bit harder because you have to think in different ways." She used a transformed mental image of a Venn diagram as well as a transformed drawn image of an empty number line as she tried to "figure a question that has that answer." The conversation between Simon and Ann shows how they used a transformed drawn image of an empty number line to support their problem-solving strategies. Simon commented that they could use empty number lines to 'see' the equations they were working on, and that, although, as Ann said, they only had to write an equation (and not "show the jumping"), drawing an empty number line meant they could be sure that they were rounding correctly and that their equation would give them the target answer.

Transformed drawn images of empty number lines during the *Create a question* activity also helped some group 1 pairs make their problems more complex. Ruth and Jill explained that, having jumped along an empty number line to make sure that the difference between 5.2 and 8 was 2.8 (their answer), they decided to make it harder by changing 5.2 to 5.23. Again using a transformed drawn image of an empty number line, they jumped 2 to get to 7.23. They mentally imaged the final jump before writing their final problem, 5.23 + ? = 8.03 (Figure 5.4).

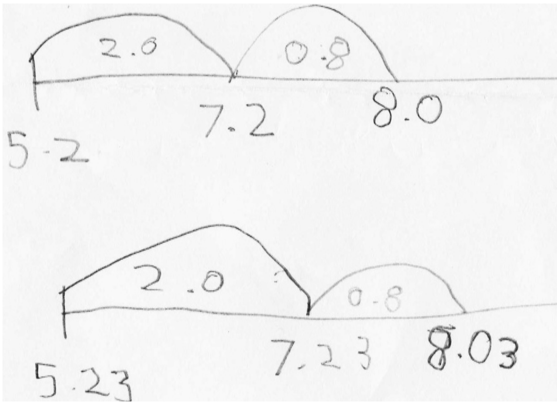


Figure 5.4: Ruth and Jill used an empty number line to make their *Create a question* problem more difficult

Imaging enabled students to develop their confidence using alternative strategies, and to modify strategies, as the above example of Jill and Ruth shows. Simon, from group 1, used two strategies to find the difference between 3.8 and 4.69, a question posed to revise the strategies of finding the difference by either subtracting or reversing and adding. He commented that he was not particularly good at reversing and adding, and needed a transformed drawn image of an empty number line to 'see' what he was doing. If he drew an empty number line, he could round to tidy numbers to solve the problem because he could 'see' what he had added. He found subtracting using place value easier, and he liked subtracting, and could subtract a little bit at a time in his head without needing to have an image. In the final assessment, Simon used number properties to solve a similar problem by reversing and adding. Asked to find the difference between 6.13 and 5.8, he wrote the equation as 5.8 + ? = 6.13. He then described his strategy: first adding 0.2 to make "six on the dot," keeping the 0.2 aside, then adding 0.13 to make 6.13, before, finally, adding the numbers together to find the missing addend (0.33). As he talked, he recorded the numbers 0.2 and 0.13 as a reminder of what he had added (Figure 5.5).

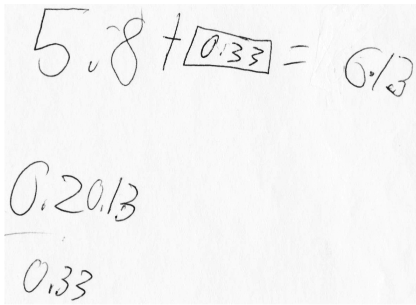


Figure 5.5: The steps Simon recorded as he used number properties to find the difference between 6.13 and 5.8

Tom used a transformed drawn image to add 523 and 246 in group 2's final assessment. He said that recording his steps on an empty number line enabled him to use his basic facts knowledge to calculate the answer, rather than jumping along in groups of 10 as he had done earlier in the unit (see section 4.2.2). He described his strategy: "it's not just a 500, it's a 5 and…just add on 2," and "if 40 + 40 equals 80, just take 2 from that."

5.2.2 How imaging is used over time

The section looks at how three students, Ruth, Jill and Simon, used imaging over the three weeks of the decimals unit and in the final assessment. In the first week of the unit, all three students referred to the physical characteristics of the materials as they imaged made decipipes (mental picture images). Ruth's description of what 0.307 would look like can be found in section 4.1.1. Simon also referred to the decipipes when he explained (to his partner) what 0.307

would look like: “It’s three-tenths like those….” Simon also pointed to made decipipes as he explained how he would add 0.5 and 0.37.

Students continued to use mental picture images during the second and third weeks, particularly when new strategies were introduced, during teacher-led discussions, or if they had found a problem more difficult than anticipated. I have already discussed Simon’s use of a mental picture image (in conjunction with number properties and materials) to help him find the difference between 2.5 and 0.89 (see section 4.1.2), and his description of how he used a mental picture image as he shuffled numbers to rewrite 0.75 + 0.6 as 0.7 + 0.65 (see section 4.1.1). Simon also used a mental picture image of made decipipes as he described his rounding and compensating strategy to add 1.09 and 0.68 (changing it to 1.1 + 0.67) (see section 4.1.1).

All three students used transformed images of empty number lines during the second and third weeks of the unit. An example of the way Ruth used a transformed drawn image of an empty number line is illustrated in Figure 5.6. Her notation suggests that, as well as using the empty number line to subtract 0.6, she was able to explain to herself how to rename one-tenth as ten-hundredths, and thus subtract the remaining 0.07.

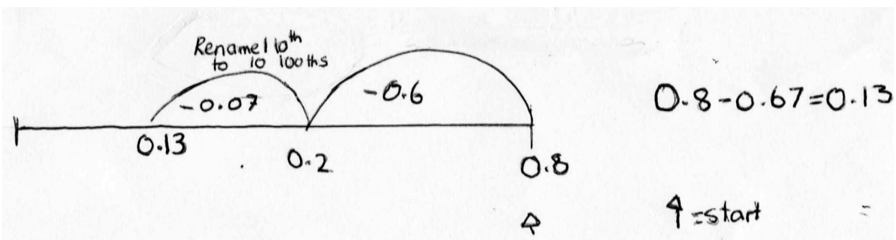


Figure 5.6: The transformed drawn image of an empty number line used by Ruth to find the difference between 0.8 and 0.67

All three used a transformed mental image of an empty number line as they found the difference between 0.7 and 0.54. Simon explained that he was able to jump back in tenths and hundredths. Ruth and Jill reversed the problem and added. Ruth commented that it was an easier problem and that she was able to work it out in her head.

Ruth and Jill seem to have preferred to use imaging (often a transformed image of an empty number line, but sometimes a mental picture image) as they solved many of the problems during these two weeks. The use of a transformed drawn image was a key part of their strategy to find the difference between 1.68 and 3.54 (see section 4.1.3). In contrast, Simon often seems to have used number properties, for example to find the difference between 0.8 and 0.67, and to solve the problems on 8 June (see section 5.3.1 for his explanation of how he added 3.15 and 2.78).

All three used transformed drawn images of empty number lines as they worked with partners on the *Create a question* activity. Simon's comments suggest that he was using the transformed drawn images to 'see' the problem, to work out how he and his partner, Ann, could write the problem, and to make sure that it was correct (for example, writing a change unknown problem that had an answer of 15.75, they chose 4.25 + ? = 20).

Ruth used a transformed drawn image, although not of an empty number line, to help her solve question 4 (6.13 – 5.8) in the final assessment (Figure 5.7). After rewriting the question as an addition problem, Ruth wrote a series of equations representing her steps as she solved it. She explained that as she wrote each step she was thinking about how she could make it up to a tidy number. Ruth solved a second problem by recording steps, and the final two problems using number properties.

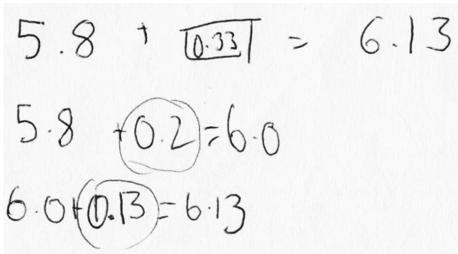


Figure 5.7: The transformed drawn image used by Ruth during the final assessment

Simon solved all four problems using number properties. He did not use transformed images; instead he recorded numbers on a piece of paper to remind him of where he was up to (see Figure 5.5). Jill used transformed drawn images of empty number lines for all four problems, two of which she solved correctly. Jill's strategy as she attempted to solve the third problem has been described in section 5.2.1.

5.3 Examples of times when students do not image

5.3.1 When problems are easy

The first problem of group 1's initial interview asked the students to find the missing addend to solve 26 + ? = 82. Jane, Ann and Simon all used number properties to solve the problem, specifically rounding to a tidy number. Ann explained her thinking:

> Add 4 to make 30. That makes an even number. Then add 50. Then plus 2. Then it would be, you'd have 50…56.

Although Ann did not use the term 'rounding,' her comment about making an even number shows that she was aware of the strategy she was using and was able to explain her working.

Problem 4 of the initial interview asked students to use multiplication and division to find a fraction of a set (3/7 of ? = 18), and was based on the activity, *Birthday cakes*, in *Book 7: Teaching fraction, decimals, and percentages*

(Ministry of Education, 2008c). Jane was the only student who was able to solve this Using Number Properties:

> J: So first you'd go 18 ÷ 3 equals 6. Then you'd go 6 x 7 which is 42.
>
> T: How did you solve that one?
>
> J: First I had to think of how many candles were on one piece of cake, so to do that you had to divide and then you times that one piece by however many pieces there were and there were 7 pieces so…yeah.

Jane had no difficulty solving the problem, and she automatically used multiplication and division to find the answer. In her explanation she related the problem to the context, and to the image of the candles and the pieces of cake, and could perhaps be said to be imaging. However, this was in response to a question that asked her to talk further about her strategy. Her initial answer was to treat the numbers as formal entities without any reference to the context, to use number properties.

Problem 1 of group 2's initial interview asked students to look for compatible numbers adding to 10 to help them find the answer. Emily and Caroline both used number properties to correctly answer the question. They looked for groups of 10, and added them together in their heads, although Emily used her fingers to cover up the numbers she had used so she could keep track of what she still had to add. The second question asked them to add 47 and 5. Emily and Caroline used the strategy that had been taught during the first term. Both explained that their first step would be to add 3 to 47 to make 50, and then add the remaining 2.

Jane moved quickly to working with number properties during the decimals unit. Following instructional activities focusing on solving subtraction problems using decomposition, students were asked to use an appropriate strategy to solve a story problem that required them to find the difference between 2.5 and 0.89. Jane recognized that she could reverse the problem and use rounding to find the missing addend. In doing so she was applying a strategy (reversing subtraction problems and adding) that had been taught the previous year as a whole number strategy, but which had not yet been introduced during this unit. When asked to talk about her strategy (she had added 1.11 to 0.89 to bring it up to 2 and then added 0.5), she said she was, "making a tidy number, just thinking add the numbers to get to the goal number. That's the only one [strategy] that automatically comes, that usually comes into mind."

The following examples of group 1 students moving straight to number properties come from week three. After instructional activities to teach the strategy of rounding and compensating to solve addition problems, the group was asked to solve 1.6 + 0.93. Some imaged or used materials, while others, including Peter and Jane, used number properties. Peter chose to solve the problem by rounding 1.6 to 2, and then adding 0.53. Jane took the three-hundredths off the 0.93 so that she could add 1.6 and 0.9. She explained that, having done this, she just had to add the 0.03 back on. It is interesting here that neither Peter nor Jane chose to use the strategy as it had been taught (rounding 0.93 to 1 and compensating by subtracting 0.07 from 2.6), and both commented that their strategies were easier for them than the taught one.

Jane and Simon used different strategies and number properties to add 3.15 and 2.78 (one of the three problems in the 8 June activity), although neither used the intended strategy of place value decomposition. Simon explained that he was "swapping numbers…because it makes it easier to plus 2.7" (solving an earlier problem, he had described this as "number shuffling" (see section 4.1.1)). To do this he "took the eight-hundredths from 2.78 and added them to

3.15 to make 3.23," and then added 2.7 to get the answer of 5.93. Jane also adjusted the problem, 'transferring' 0.02 from 3.15 and adding it on to 2.78 to make the equation 2.8 + 3.13. She was then able to solve the problem in one step. During a later interview, she explained her choice of strategy, saying that she was "trying to get rid of things like the hundredths and thousandths" because they are "kind of annoying" and make it "harder to add all in one go."

In group 2, Emily tended to move quickly to manipulating just the numbers. A teaching focus for this group was to encourage students to add or subtract from a number (for example, 56 + 31 as 56 + 30 and then 86 + 1) rather than separating both numbers into tens and ones (for example, 56 + 31 as 50 + 30 and 6 + 1). Two of the learning outcomes asked students to solve problems involving adding or subtracting two-digit numbers without renaming. The following examples show the different ways Emily approached these problems. In the first (during week 1), she was asked to add 31 and 27. Students were able to image made bundles of ice cream sticks (mental picture images). Emily used number properties, explaining that she had added 2 and 3 and "that equalled 50" and then the 1 and the 7. To solve the second problem (64 - 31), Emily subtracted from the first number. She explained she "minused" the 30 to make 34 and then the 1 so the answer was 33. The third problem asked students to use number properties to add two three-digit numbers (361 and 423). Emily recorded just the answer. Her strategy was to separate the hundreds, tens and ones: 300 + 400 = 700, 60 + 20 = 80, and 1 + 3 = 4, giving an answer of 784.

5.3.2 An example of a non-imager

Peter does not appear to have used imaging, except during teacher-led activities that included specific questions designed to encourage and support students to use imaging. Peter attempted to solve question 3 in the initial interview (4 ÷ 5) using number properties. He multiplied 4 by 5, and then "cut" 20 into 4 to get an answer of 5. It was suggested that it might help if he used a drawn picture image. Peter drew a diagram with 4 rectangles, but was still not able to answer the question correctly. He was asked whether drawing helped him to explain what he was trying to do, to which he answered "no."

Peter had a very good understanding of decimal place value at the start of the unit: he explained the relationship between tenths, hundredths and thousandths in terms of multiplying or dividing by 10. He was one of the students who moved quickly to Using Number Properties. Generally, Peter solved problems correctly, often using the strategy of rounding to tidy numbers. Two examples illustrate his strategies for solving problems. In the first, Peter added 1.6 and 0.93. He rounded 1.6 to 2 and explained that it was easy to add the rest (0.53). Peter also chose to round to solve the second problem, finding the difference between 0.8 and 0.67. He explained his working: "I rounded 0.8 to 1 whole. Added 0.33 to 0.67 to get to a tidy number." He then subtracted 0.2 from 0.33 to get the answer of 0.13.

In the final assessment, Peter answered two of the four questions correctly. He gave two answers to question 1 before finally answering it correctly (with teacher support so it was not marked as correct). Initially, he used number properties and total place value decomposition (the whole strategy is included here to show his thinking, although it is quite difficult to follow):

> First I would go 3 + 1 = 4, so just adding them together. Then I'd probably go 1 + 6 because that's not above 20, and that would equal 7, and then I'd just go 1 + 9 equals another whole. Then you'd have one left so it would be 4 and, no I'd add the 7 and the 1, which would make 8, and then you would have 5.8.

At my suggestion, Peter recorded his steps. He used the same strategy, and his second answer was also incorrect (Figure 5.8).

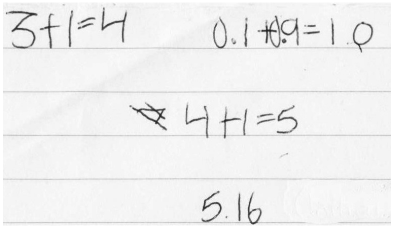


Figure 5.8: Peter recorded his steps as he made a second attempt to add 3.21 and 1.96

It was only after it was suggested that he try rounding that he was able to solve the problem correctly, again recording his steps. Although Peter recorded his steps as he attempted to answer, he does not appear to have used imaging, even when told his answers were incorrect (unlike Ruth who used the same method of recording in the final assessment (see section 5.2.2)). His explanations suggest he was attempting to use number properties. He commented that rounding was his "strong point" when it came to solving problems, although, interestingly, it had not been his initial choice of strategy to solve this problem. He also commented that, when it came to solving problems, he was not always accurate ("I get like one off").

Chapters 4 and 5 have identified two distinct forms of imaging – picture and transformed imaging, and within each of these mental and drawn images. The chapters have also described the ways in which imaging is used by students as they move towards Using Number Properties to solve problems. These findings relating to both types of imaging, and the way imaging is used by students, will be discussed in the following chapter.

Chapter 6: Discussion

6.1 Introduction

This chapter discusses findings from the previous two chapters. The findings will be discussed in six sections:

- Picture images of the materials used in teacher-led activities;
- The use of transformed images;
- Students' preference for drawn rather than mental images;
- The pathway between the phases of the teaching model;
- Imaging as part of a climbing wall to Using Number Properties;
- The role of imaging in linking Using Materials and Using Number Properties.

To refresh the explanations of images that I am using:

- Students used picture images when they imaged the materials utilized to make problems during teacher-led activities.
- Mental picture images were created by students describing how they would manipulate imagined or shielded materials, or materials they could see but not touch.
- Drawn picture images are drawings of the materials used to make the problem.
- Students used transformed images when, rather than imaging the materials used to make the problem, they selected an unrelated image (mental or drawn) to help them solve the problem. Note that the transformed image that was used most often during this study was that of an empty number line.

In this chapter I introduce the term *mathematical resources* to describe the phases of my model that students are using as they work to solve problems. In using a particular mathematical resource, for example, materials, students would then choose the specific materials to help them solve the problem.

6.2 Picture images of the materials used in teacher-led activities

The first finding relates to *when* and *how* students used picture images, as well as to the language students used to describe these images.

The use of picture images was promoted by me as part of instructional teaching activities, for example, as students moved from the Using Materials to Using Imaging phase of the STM, or if a new piece of equipment was introduced. This was to enable students to develop particular representations of the concept and knowledge of the strategy being taught (Martin, 2008; Ministry of Education, 2008b). Students were asked to describe what they would do if able to manipulate either imagined or seen materials.

6.2.1 When picture images were used

Picture images were most often used during teacher-led activities. Students also used picture images during the initial interviews, and some, notably David, used picture images when solving problems independently.

The picture images used by students during teacher-led activities were generally mental picture images. Students described how they would solve problems by manipulating either imaginary or shielded materials, or materials they could see but not touch in response to anticipatory questions. On these occasions students did not have a choice of the type of image they used (although one or two chose not to image at all, but moved straight to Using

Number Properties); they were asked by me to describe mental picture images of specific materials.

Picture images (both mental and drawn) were also used by students as they moved from unsuccessfully attempting to use number properties to using imaging to solve problems during the initial interviews. Both the mental and drawn picture images replicated the materials that had been used by students to make the problems and by the teacher to introduce them during earlier lessons. For example, Emily and Tom described mental images of hands, which they used to count on, and David used drawn circles to find fractions of sets. Jane and Simon mentally imaged and drew rectangles representing chocolate bars as they attempted to solve a division problem where the answer was a fraction. It should be noted that the use of materials was not offered to students during the initial or final interviews. It is possible that their choices may have been different if they had been able to use materials.

Picture images were therefore one of the teaching model phases students used to solve problems. They were often used during teacher-directed activities, when focusing questions were asked, and strategies demonstrated to support students' progress between the phases of the STM. Students also reported (or were observed) using picture images (both mental and drawn) as they worked to solve problems, as can be illustrated by the examples in section 4.1.2. On these occasions, picture images were often used in conjunction with other teaching model phases, for example, transformed images or materials. The ways in which students used these picture images will be discussed in the following section.

6.2.2 How picture images were used

Picture images were used both to support students' developing understanding, and as part of the teaching model phases they selected from when solving problems.

Picture images, particularly mental picture images, seem to have been important in helping group 1 students develop an understanding of both decimal place value and the strategies being taught during the early stages of the unit. A number of students (including Ann, see section 5.1.2) commented that imaging had helped them understand decimal place value because they were able to 'see' the relative sizes of the numbers. Here the use of picture images is similar to Pirie and Kieren's Image Making stage, during which "learners work at tasks, mental or physical, that are intended to foster some initial or extended conceptions for the topic to be explored" (Pirie & Martin, 2000, p. 130). In contrast to group 1, only one group 2 student, Tom, described mental picture images as important in helping him understand the composition of the numbers he was making. This may be because the numbers (tens and ones) were familiar to group 2 students, whereas decimal place value was new to those in group 1 (although they would have previously encountered decimal numbers, they had not investigated their meaning). Students in both groups, however, used imaging in a way similar to the P-K Image Making stage, to develop and extend their understanding of the strategies being explored. Simon's use of a mental picture image (in conjunction with materials to find the difference between 2.5 and 0.89) to extend his understanding of both decimal place value and the strategy, subtraction using decomposition, is an example of this. This use of picture images seems to have been a feature of students' developing knowledge about the strategies being taught.

Picture images (usually mental rather than drawn) were one of the teaching model phases students used as they worked to solve problems, especially during the early part of the teaching sequence for each strategy. Their descriptions of their mental picture images seem to correspond to those depicted by Hughes (2002), that Using Imaging is the visual imagery of absent objects and the creation of picture images. Students in this study used mental picture images as they moved between the phases of the STM. However, they did not necessarily move through the phases of the STM in the linear, sequential way suggested by the diagram or explanation in *Book 3: Getting started* (Ministry of Education, 2008b). Ruth's use of a mental picture image as she tried to add 1.6 and 0.93 by rounding and compensating illustrates this finding (see section 4.1.1). While most students used mental picture images, David often used drawn picture images to solve problems, frequently in combination with transformed images or materials. He did this throughout the three weeks of the unit, and also in the final assessment interview. The students' descriptions of how they would manipulate the picture images reflected the actions they would have performed on materials. This will be discussed below.

6.2.3 The language used by students as they described manipulating picture images

The language used in students' descriptions of both mental and drawn picture images appears to be closely connected to the physical characteristics of the materials and the context of the problem. During the initial interviews, the students who drew circles to represent the birthday cake or rectangles to represent the chocolate bars talked about candles, cutting or splitting the cake, eating the cake, the number of pieces of cake, the pieces of the chocolate bar, and how it was to be shared between the five people (see Figures 4.3 and 4.4).

This close connection between the picture image and the characteristics of the materials also occurred during the early stages of the decimals unit. Students pointed to the decipipes, used their hands to represent the size of the decipipes, or described their physical characteristics. For example, in her description of what 0.307 would look like on a decipipe, Ruth referred to the thousandths as the "silver round ones." Group 1 students also talked about "colouring in" pieces of decimats as they described adding two decimals together. In contrast, group 2's descriptions of their images only seemed to be closely linked to the physical properties of the materials when a question was phrased in a way that required them do so. An example of this is Emily's response to a question asking what her favourite materials would look like if she were to make 45: "4 jellybean packets and 5 single jellybeans by itself." Generally, group 2 students talked about how many tens and ones there were in a number. This may have been because they were very familiar with the materials used to make and solve problems (which would have been used in previous years), whereas the decipipes and decimats (as well as the decimal place value concepts) were new to the group 1 students. This conclusion is supported by the language and gestures group 2 students used when two-digit addition problems were modelled and solved using a hundreds board. They pointed to the hundreds board as they created mental picture images of the steps they would take to add the two numbers, or talked about jumping across or down. Although the hundreds board was familiar to all the students in the group, it had probably not previously been used in this way.

Gray and Pitta (1999b) discuss the close relationship between an image and the language used to describe it. They describe this type of image as being "*essential for thought* in that they guide the use of a procedure" (p. 14). In their study they report students describing specific, detailed images (often including detailed properties of the materials, for example, colour) reminiscent of the actions they would have performed if they had been using materials (Pitta & Gray, 1999b). The language used, at least initially by students in this study, can be described as 'essential for thought;' it was closely related to both the materials and the actions that would have been performed using them. Further, the use of the image was an essential element in the students' abilities to solve the problem.

The language used by students to describe their actions on mental picture images changed during the course of the decimals unit, in particular. Instead of referring to the physical characteristics of decipipes, students described manipulating tenths, hundredths and thousandths. An example of this change over time is Simon's description of how he would shuffle five-hundredths from the decipipe representing 0.75 on to that representing 0.6 (see also section 4.1.1). Simon does not make any reference to the physical characteristics of the pipes; instead, he talks about tenths and hundredths. However, although the language is more abstract in terms of the physical characteristics of the materials, Simon's description is still closely related to the way he would have manipulated the materials; the image is still "essential for thought" (Gray & Pitta, 1999b, p. 14).

It is noteworthy that, in this study, students with a wide range of abilities described manipulating images that were closely related to actions they would have performed on materials. One explanation for this could be the structure of the Using Imaging phase of the STM, with its emphasis on describing how absent or screened materials might be manipulated to solve a problem (the picture images described by Hughes, 2002). It may also be that, for some students, this is a necessary point on their developmental pathway. This finding represents a notable difference to those reported by Gray and Pitta

(1999b), who found that the students who described these specific, detailed images were low achieving.

Another feature of the use of picture images in this study is that they appear to be only part of students' repertoire of images, and that there seems to be a progression from using picture to using transformed images.

6.3 The use of transformed images

A second finding relates to students' use of transformed mental and drawn images. This section discusses the ways in which transformed images were used by students, including the use of transformed images as thinking tools and as a means of representing mathematical solutions, and the language used by students when describing transformed images.

6.3.1 What are transformed images?

Students in this study appear to have moved from using picture images to transformed images as part of a progression towards Using Number Properties. I have described these images as transformed for two reasons. First, they were not related to the materials introduced to students during teacher-led activities, or to those students used to make or image problems during teacher-led activities in the Using Materials or Using Imaging phases. Second, the images supported the development and transformation of students' thinking. In particular, they gave students the opportunity to develop their own solutions, and, at times, fostered the development of more sophisticated strategies; in this way supporting the development of knowledge needed to use number properties (Klein et al., 1998). The most frequently used transformed image was an empty number line, although Jane (in group 1) described using a mental image of a Venn diagram.

Students may have chosen to use an empty number line because it had been used extensively in class activities; it was a didactical model (Klein et al., 1998). Teachers often modelled the use of the empty number line by using it to record students' strategies and thinking in modelling books. Indeed, members of group 2 explained that their teacher had taught them to use an empty number line in Term 1. During the course of both units, I used empty number lines to record students' explanations of their strategies in the groups' modelling books. At times I also asked them to record their strategies by creating empty number lines. Jane, the student who said she preferred to use a mental image of a Venn diagram, explained her reasons for choosing this image: her sister had shown her how to use a Venn diagram to organize and store information.

6.3.2 How transformed images were used

This section discusses findings related to the way students used transformed images as thinking tools and as a way of representing mathematical solutions. Most students, once they had developed a notion of the meaning of the numbers they were using (for example, the place value of the decimal numbers), appear to have stopped using picture images. Some (for example, Jane and Peter in group 1) moved directly to Using Number Properties in the way suggested by Hughes in his revised teaching model (2008, personal communication from numeracy advisers; see also Appendix F). In this, Hughes recognized that not all students move systematically through the three phases of the STM. Rather, after a strategy has been introduced using materials, some students move straight to Using Number Properties. In fact, there are a number of examples where Jane and Peter did not use imaging at all, except during teacher-led activities involving direct questioning. However, other students, such as Ruth and Jill, appear to have chosen a transitional step, a transformed image, to link what Pirie has described as a "disconnected mental leap" between imaging and abstract number properties (2002, p. 930). In constructing transformed images, students appear to have used their knowledge of the picture images, particularly the way picture images could be manipulated to solve problems. These transformed images were then used by students like Ruth and Jill to help them make connections between picture images and the formal representations of the problems.

Transformed images as thinking tools

Transformed images seem to have been used by students as thinking tools. Beishuizen (2010), in his study of the empty number line, found that one of its key features is an ability to enhance flexible thinking, and enable students to move to more efficient and flexible ways of solving problems. As well as a flexible thinking tool, students in this study seem to have used transformed images as generalized thinking tools. I have described these as generalized because the same transformed image could be used in a variety of situations and for a variety of tasks. For example, most students in both groups 1 and 2, who used a transformed image, chose an empty number line irrespective of the differences in the numbers used and the types of problems being solved.

When students used the empty number line or another transformed image, they were no longer using images related to particular actions on specific materials (Pirie & Kieren, 1989, 1992). Rather, they were able to explore mathematical solutions and, through 'progressive mathematization,' develop formal and more efficient strategies for solving problems (Beishuizen, 2010; Klein et al., 1998). Using an empty number line helped Tom use his basic facts' knowledge, rather than jumping in groups of ten and then ones, during a problem posed in the course of group 2's final assessment. Ruth and Jill used an empty number line to explore possible mathematical solutions to the problem asking them to find the difference between 1.68 and 3.54. Like Tom, they developed a more efficient solution – in this case jumping beyond the number and then subtracting to get the correct answer, a strategy that had not been taught during the instructional activities. In both examples, the students solved the problems by operating on the numbers, although they were not yet at the stage of Using Number Properties.

There are a number of examples of students using transformed images to solve problems when either the numbers or the problems were difficult. The use of transformed drawn images of empty number lines by group 1 students during the *Create a question* activity has been discussed in section 4.1.2. Similarly, Emily used a drawn image of an empty number line to solve some of the problems in the final assessment because she felt the addition or subtraction of three-digit numbers made the problems too tricky for her to solve in her head (see section 5.2.1). Simon used a drawn image of an empty number line to solve a problem where he was asked to reverse a subtraction problem and add. He said he had used an empty number line because he was not confident reversing subtraction problems and adding. The empty number line meant he could 'see' what he was doing, particularly what he needed to add as he attempted to round to tidy numbers. He added that, if he had been solving the problem using subtraction, he would have been able to solve it easily in his head, because he was confident he could find the correct answer by subtracting in small steps (see section 5.2.1).

The transformed images discussed in this section seem similar to the Pirie and Kieren 'Image Having' stage (Pirie & Kieren, 1989, 1992). Here they describe students' problem solving as no longer dependent on the need to perform particular actions on objects. Rather, the "image itself, as a mental object, can be used in mathematical knowing" (Pirie & Kieren, 1992, p. 247).

Transformed images (particularly drawn empty number lines) appear to have acted as constructivist tools, as part of a self-regulating process students used to solve problems. I would suggest that, in using transformed images, students were demonstrating the attributes characterized by the 'Thinking' Key Competency in the *New Zealand Curriculum* (Ministry of Education, 2007a). In particular, transformed images supported the "creative, critical, and metacognitive processes" students used to make sense of the ideas, and to actively seek, use and create knowledge (Ministry of Education, 2007a, p. 12). Students built on their existing knowledge and used it to create the new knowledge needed to solve a problem (Beishuizen, 2010; Ell et al., 2010; Klein et al., 1998; von Glasersfeld, 2000). In doing so, it seems, as Beishuizen (2010) claimed, that students were able to use the empty number line to enhance the flexibility of their mental thinking and help them move towards more efficient and flexible ways of solving problems.

Transformed images as a way to represent mathematical solutions

Students used transformed images, particularly drawn images of empty number lines, to represent mathematical solutions, to facilitate the development of these solutions, and to communicate, discuss and modify their ideas.

Ruth and Jill's use of an empty number line to find the difference between 1.68 and 3.54 has been discussed in section 4.1.3. The transcript of this discussion includes Ruth reminding Jill that the problem was not an addition equation, discussions about how best to solve the problem and which numbers to add or subtract, and reminders about what they still had to do. As both Ruth and Jill contributed actively to the discussion, they developed a shared meaning that supported the development of their mathematical understanding and helped them to actively build knowledge (Brophy, 2006; Cobb, 2000; von Glasersfeld, 1993).

6.3.3 The language used by students when describing transformed images

I have discussed the relationship between the language used by students and the physical attributes of picture images. There was a noticeable change in the vocabulary students used when they described manipulating transformed images. They no longer made reference to the context of the problem or to the characteristics of the materials. The students did not refer to particular actions that might have been carried out on the materials as they had, for example, when describing how they would share the birthday cake to find a fraction of a set. Instead, the language students used illustrated their active use of transformed images. They described jumping along an empty number line, jumping too far and then jumping back, using big and little jumps, and adding the jumps together to find the answer. The students also used more formal mathematical language. Group 2 students talked about "minusing," "taking away," or "counting up." Students in group 1, in particular, talked about tidy numbers, and how they could round to tidy numbers. In their explanations of their solutions, students in both groups described how they had used both known and derived facts, although they did not refer to them using this mathematical language. Once students, particularly group 1, had determined how to rewrite a story problem as an equation, they made no further reference to the context of the problem unless it was required to explain their solution (for example, comparing runners' times or the distance of throws). This is in contrast to the initial interviews, especially those involving group 1, when students often referred to the chocolate bars, the people or the birthday cake.

Gray et al. (2000) comment that some students have the ability to focus on detail relevant for that moment, and make choices and filter out irrelevant detail, such as the nature of the materials or the context of the problem. This seems to have been what students in this study were doing as they manipulated transformed images of empty number lines. However, in contrast to Gray et al.'s (2000) findings that this ability to filter out irrelevant information was limited to high achieving students, students in both groups in this study successfully used transformed images in this way.

6.4 Students' preference for drawn rather than mental images

Most students preferred to use drawn rather than mental images. In reaching this conclusion, I have included data from observations of the choices students made, as well as that from discussions with students about their choices. Drawn images include both picture and transformed images. They are the diagrams or numbers recorded by students, for example, David's drawings of decipipes or empty number lines. Mental images can also be either transformed or picture images, created when students describe how they would manipulate imagined or screened materials, or when they look at, but do not manipulate, made materials.

Section 6.4.1 considers the types of drawn images used by students. Students' reasons for preferring drawn images are discussed, including the importance of drawn images in helping students 'see' what the problem was asking them to do, as a way of keeping track of what they were doing, and helping them focus on the calculations associated with the problem. This section also examines the use of drawn images to support 'progressive mathematization,' and as a way for students to revisit and reflect on decisions they had made. Students' use of mental images is discussed in section 6.4.2.

6.4.1 Students' choices of drawn images

The drawn images most often used by students during the two teaching units were of empty number lines. One student, David, also frequently drew representations of the decipipes during the decimals unit, and a number drew representations of birthday cakes or chocolate bars during the initial interviews. Students gave a number of reasons for preferring drawn images. First, they said that drawn images helped them to 'see' what the problem was asking them to do. Second, drawn images meant they could keep track of what they had done and still had to do. Third, using a drawn image meant students could concentrate on the next computation, without having to remember all their previous steps. They had a physical reminder of what they had done if they were distracted (often a reality in a classroom). Finally, students observed that drawn images enabled them to revisit the strategy they had used to solve a problem, and check that both their use of the strategy and their computation were correct.

Drawn images to help 'see' what the problem was asking them to do

Students used drawn images (both transformed and picture) to define the scope of problems, highlight possible solutions, and see how they could use their existing knowledge as they attempted to find a solution. David often used drawn picture images. He commented that the diagrams of decipipes that he drew and labelled as he solved problems gave him a "clear picture" of what the problem was asking him to do, and how he could solve the problem using his decimal place value knowledge.

Drawn images to keep track of what students were doing and as a record of thinking

Drawn images, particularly empty number lines, provided a stable record of what students had done and still had to do, enabling them to keep track of their strategies as they solved problems. This also meant that students were able to make decisions about whether a strategy was working or likely to work, or whether they needed to try something else, as the example of Ruth and Jill discussed in section 4.1.3 illustrates. One effect of this was to reduce the load on their working memory, which, as Klein et al. (1998) note, is one of the features of the empty number line. Ruth and Jill adapted the empty number line to suit their purpose by continuing to move to the right to successfully solve the subtraction phase of a number of problems, rather than hopping back to the left. It appears they were using the empty number line as a constructivist tool, to actively build knowledge, and adapting it to their needs.

In addition to helping students keep track of their steps as they solved problems, drawn images provided records of their thinking that did not change. A number of students commented that, if they only used a mental image, it was difficult to remember all their steps as they worked to solve a problem. They found this frustrating because, if they forgot what they had done, they had to start again. In contrast, when they used a drawn image they could go back and check, and, if necessary, change or correct their solution. Tom highlighted another classroom reality, distractions, in his comments about why he found drawn images helpful. He said recording his thinking on paper meant he could remember what he had done even if his he mind wandered on to another topic, like an invitation to a friend's birthday party.

Drawn images to help students focus on computation tasks

A number of students reported that a benefit of using an empty number line was that they were able to focus on computation tasks within a problem without forgetting what they had already done. For Jill using an empty number line meant that she could concentrate on using her basic facts and place value knowledge to solve each step of the problem, without having to worry about remembering what she had already done. Emily used an empty number line to help her find the missing addend to solve 47 + ? = 123 by using derived facts (see section 4.2.1).

Drawn images to support 'progressive mathematization'

There are a number of examples of drawn images (generally empty number lines) being used by students to develop more efficient and flexible ways of solving problems (Beishuizen, 2010). Tom often used a drawn or mental image of empty number line. During the teaching unit, he talked about jumping along an empty number line in groups of ten. However, he used a different strategy to solve one of the problems during final assessment. Rather than jumping in tens or multiples of ten, he explained that, this time, he had been able to use basic and derived facts to add 523 and 246, and that this was because he had recorded his jumps on a drawn empty number line (see section 5.2.1). It appears that using the number line reduced his memory load (Gray et al., 1999; Klein et al., 1998). As a result, Tom used a more sophisticated and abstract strategy, and one that was more like formal mathematics, a process described by Beishuizen (2010) and Klein et al. (1998) as 'progressive mathematization.'

Drawn images as a tool to revisit decisions

Drawn images meant students could revisit the mental decisions they had taken while solving a problem. At times they used the drawn image to confirm they had solved a problem correctly, for example, by checking their addition or subtraction was accurate. The benefit of being able to do this was explained clearly by Simon. He said using an empty number line enabled him to go back and see the problem again, to "check it…go over and see if the corrections are right." There were other occasions where, by revisiting a mental decision, students were able to adjust their strategy while still working on the problem. The clearest illustration of this is the decisions Jill and Ruth made as they worked to find the difference between 1.68 and 3.54. As they recorded their jumps on an empty number line, they came to the conclusion that it would be too difficult to add the jumps together to get the answer. As a result they decided to try an alternative strategy, which has been described in section 4.1.3. Students also used drawn images to work out where an error had been made, as Jill explained during the final assessment interview:

> I can see what I've done and I can remember, because if I get confused I can go back and figure out what I've done wrong as well.

6.4.2 Students' use of mental images

Students' preference for drawn images does not mean that they did not use mental images. The greatest use of mental imaging was during instructional teaching activities, as students moved from Using Materials to Using Imaging. Students were actively encouraged to create mental picture images in response to my questions about how they would manipulate materials (either imagined or visible) if they were able to touch the materials. This followed the progression outlined in the Strategy Teaching Model (Ministry of Education, 2008b).

Students also used mental picture images to help them 'see' problems, and to check that they had solved them correctly, including Ann, who used mental images in both these ways during the early stages of the decimals unit. She commented that imaging the decipipes helped her see the "tenths and stuff," and develop her understanding of decimal place value. During the *Make 0.5* activity, Ann checked the problem she had written was correct using a combination of materials and a mental picture image to, first, partially make the problem on decipipes, and then image the rest of it.

A small number of students expressed a preference for transformed mental as opposed to drawn transformed images. Jane said she generally did not need to use an image, but preferred to manipulate a mental image if she needed to use something other than number properties. Her chosen image was a transformed mental image of a Venn diagram (see Figure 4.11). She compared her mental image of a Venn diagram to a cabinet; somewhere she could store numbers and then get them out when she needed them. In using a Venn diagram, Jane was "working with metaphors" (Pirie & Kieren, 1994a, p. 40). For her the mathematics was the image of the Venn diagram and she was working with that image. She explained that, although her exact use of the three parts of the Venn varied depending on the problem, she generally used the central part to store numbers while she manipulated other numbers in the outer circles.

At times there was a continual interplay between drawn and mental images, and indeed between all the teaching model phases available. Students might start to solve a problem using a transformed drawn image and then decide to use a mental picture image to solve part or all of the rest, usually because the problem was more difficult than they had anticipated.

6.5 The pathway between the phases of the teaching model

Students in this study used four phases of imaging (mental picture images, drawn picture images, transformed mental images and transformed drawn images) as they worked to solve problems. Data collected during the study indicates that the students moved between these four imaging phases and using materials and using number properties in a fluid way, both in terms of their chosen starting points when asked to solve a problem and the pathway they followed as they attempted to solve it. This contrasts to the progression suggested by the two-headed arrows and the linear Strategy Teaching Model. The STM model describes students moving backwards and forwards between Using Materials and Using Imaging or between Using Imaging and Using Number Properties until they are able to solve problems using only number properties and a strategy becomes part of their strategy repertoire (Ministry of Education, 2008b).

In using the metaphor of a pathway to describe the steps taken by student, I am adapting Pirie and Kieren's phrase, "paths of growth of mathematical understanding" (1994b, p. 186), which they used to represent the ways in which students move between the layers of their theoretical model in order to develop understanding. I have used the metaphor of a pathway to describe the students' movement between the six phases (four imaging phases and materials and number properties phases) as they worked to solve single problems, and as they solved a range of problems involving the same strategy over the course of the units. These six phases are collectively described as mathematical resources. It should be noted, however, that references to the Using Materials or Using Imaging phases are to the phases defined in the STM, and references to phases of the STM are to the three phases of that model (Ministry of Education, 2008b). This is because I was following the progression outlined in the STM during both groups' units.

6.5.1 Students' starting points

There are a number of examples of students choosing as their starting point a mathematical resource that was different to the one that I had suggested they use. Most commonly, students, including Caroline (group 2) and David (group 1), chose to use materials or imaging (including mental picture, drawn picture and transformed drawn images) although they had been asked to use another phase of the STM (usually either Using Imaging or Using Number Properties). Others, particularly Peter and Jane from group 1, chose to use number properties rather than making the problem with materials or imaging it. In describing this finding, I am inferring that students evaluated the complexity of the problem and chose the mathematical resource appropriate to their needs and current knowledge. Caroline and David often lacked confidence when solving problems. Their use of, usually, mental or drawn picture images or materials may have helped them underpin their current knowledge and connect this with the next step on their pathway, as well as achieving success by solving the problem. In choosing a mathematical resource other than the one I had suggested, it could be argued that these students were using critical thinking processes to make sense of what they were being asked to do, that they were actively seeking and creating knowledge using the 'Thinking' Key Competency of the *NZC* (Ministry of Education, 2007a). Students' active selection of their starting points within the mathematical resources can also be seen in activities where they were given a choice of resource to begin with. Again, it appears that, in choosing to start with different mathematical resources, students were using critical thinking processes to select the one most appropriate to their current knowledge. This can be illustrated by the different mathematical resources chosen by Emily, Caroline and Tom (group 2) to solve problems in the *Reversing numbers* activity: Emily used number properties, Caroline used materials and made the problems with animal strips, and Tom created a transformed drawn image of an empty number line.

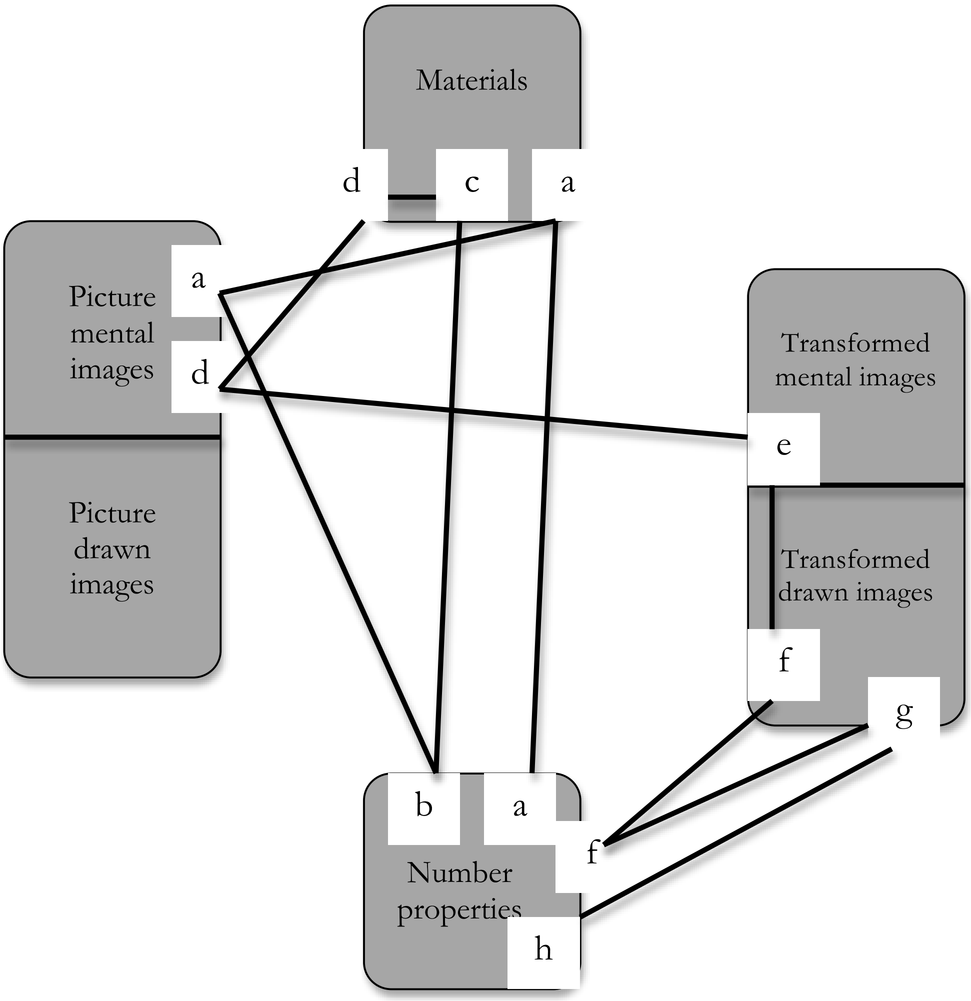
6.5.2 Students' pathways when faced with difficulties while solving a single problem

When faced with difficult problems (sometimes because they were more complex than initially thought) students used a variety of mathematical resources in their attempts to solve them. These might include using materials, one of the four imaging phases or number properties to solve all stages of a problem. At other times students chose, for example, to solve part of the problem using materials, and then created a mental picture image to solve the remainder. There are examples of students (for example, Simon in section 4.1.2) starting to solve a problem using number properties, and then moving to materials or an imaging phase because the problem was more difficult than anticipated. In fact Simon used both materials and a mental picture image when he corrected his error and when he developed his incomplete understanding of decimal place value. Students also moved between the different imaging phases. For example, Ruth started to solve a problem using a transformed drawn image, but then used a mental picture image, by looking at materials over another students' shoulder, to rename tenths as hundredths (see section 4.1.1). At times, students' decisions about which mathematical resource to use were made independently, although at other times they were made with my support.

In discussing the way students approached problems they found difficult, I have discussed students moving between mathematical resources rather than 'folding back,' which is the phrase used in the STM and by Pirie and Kieren (Ministry of Education, 2007a: Pirie & Kieren, 1989, 1992, 1994a, 1994b). Students 'fold back' to a previous phase of the STM to "connect mathematical abstraction with the actions on materials" (Ministry of Education, 2008b, p. 5). Pirie and Kieren (1992) also describe students' 'folding back' if a problem is not immediately solvable or to extend their current inadequate level of understanding. The concept of 'folding back,' as used by both Pirie and Kieren and the STM, seems to imply a deliberate act on the part of the students. In contrast, the analysis of the students I observed during this study suggests that they 'moved around' a variety of mathematical resources, seemingly without deliberately choosing a particular mathematical resource. Instead students selected the mathematical resource that best suited their needs at a particular time. It is for this reason that I have chosen not to use the term, 'folding back.'

6.5.3 Students' pathways over time

Students used a variety of mathematical resources as they worked on problems over the course of the teaching unit. Figure 6.1 illustrates the varied pathway Simon took as he solved problems asking him to find the difference between two decimal numbers. Initially, all students used materials to make problems involving place value decomposition, although this is not shown in Figure 6.1.



a	Day 5: finding the difference between 2.5 and 0.89 (as a subtraction problem) – Simon initially attempted to solve the problem using number properties
b	Day 6: finding the difference between 0.8 and 0.67 (as a subtraction problem)
c	Day 6: finding the difference between 0.66 and 0.9 (reversing and adding)
d	Day 6: finding the difference between 0.8 and 1.6 (reversing and adding) – Simon starts with decimats but moves to decipipes
e	Day 6: finding the difference between 0.7 and 0.54
f	Day 7: finding the difference between 3.8 and 4.69 – if reversing used a transformed drawn image, if subtracting using place value decomposition using number properties
g	Day 11: *Create a question* – writing a question to match a specific answer
h	Final assessment: finding the difference between 6.13 and 5.8

Figure 6.1: The fluid pathway followed by Simon

6.5.4 Fluid pathways supporting developing knowledge

The pathways taken by students in this study, when they solved single problems and over time, appear more complex than the movement between adjacent phases described by the two-way arrows in the STM (Ministry of Education, 2008b). They seem closer to Pirie and Kieren's description of their theory of growth of understanding as a "continuous path traced back and forth through levels of knowing" (1994b, p. 188). In their analysis of the development of Sandy's understanding, Pirie and Kieren (1992) map his growing understanding (see p. 37 for a diagrammatic representation of the P-K theory). Having started by Image Making, Sandy moves through successive stages of the model to Formalizing as he solves a variety of problems. The next problem causes him to return to Image Having before moving first Property Noticing and then Formalizing and he solves successive problems.

In developing their own pathways, students in this study appear to have been actively building knowledge (Ministry of Education, 2007a; von Glasersfeld, 1993). This can be illustrated by the representation of Simon's pathway in Figure 6.1. The importance of fluid, non-linear pathways is emphasized by Pirie and Kieren, who state that, "every student will have a singular path for any topic," which enables them to "re-member and re-construct new understanding" (1994b, pp. 186, 188).

6.6 Imaging as part of a climbing wall to Using Number Properties

A sixth finding relates to the way imaging was used as a scaffold by students as they worked towards the acquisition of number properties. The imaging phases appear to provide students with the confidence to attempt problems they may otherwise have decided were too difficult, and to provide an alternative when the pathway to number properties was blocked by incomplete understanding.

I have chosen to use a metaphor of a climbing wall to describe the mathematical processes used by students as they moved towards the acquisition of number properties. The STM uses the metaphor of a 'bridge' to describe the role of Using Imaging as the link between Using Materials and Using Number Properties (Ministry of Education, 2008b). However, the pathways followed by students in this study were more complex and fluid than the linear backwards and forwards pathway implied by the use of the word 'bridge.' I will briefly describe key features of a climbing wall, before discussing the role of imaging in helping students move towards number properties.

A student approaching a climbing wall chooses, first, an appropriate starting point, and then looks to see how to reach the top. Most students will attempt to climb a wall in as direct a way as possible, linking one foot- or handhold with the next. However, during their ascent they may encounter unexpected difficulties, causing them to move sideways or downwards, reaching for a suitable foot- or handhold (or any combination of the two) before finding another pathway. This may happen several times before they ultimately achieve their goal. In this metaphor, the top of the climbing wall is number properties, and the lower sections of it, materials. The rest of the wall represents the imaging phases. Students choose the most appropriate path. It does not matter whether they need to retrace their steps or move sideways as long as they get to the top of the wall. This is how I believe students approach mathematical problems; they choose the most appropriate mathematical resource to tackle a task, and change mathematical resources as many times as they need to in order to achieve the abstract understanding of number properties.

6.6.1 Imaging as a hand- or foothold giving students the confidence to have a go at solving a difficult problem

In both teaching units, students used imaging (often transformed drawn images of empty number lines) to help them tackle problems they perceived as difficult. All the students in group 1 used transformed drawn images of empty number lines to help them write questions to match a specific answer during the *Create a question* activity. Asked why she had used a drawn image, rather than number properties as she often did, Jane said it was because the problem asked them to think in different ways and that it was harder to "figure out questions that had that answer" than just finding the answer to a question. In this activity, it appears that the drawn images were essential tools in helping students to both build mathematical meaning and develop deeper mathematical understanding.

6.6.2 Imaging when the path to number properties is blocked by incomplete understanding

Students' use of imaging as part of the pathway towards using abstract number properties has been discussed in section 6.5. This section looks specifically at how students used imaging to attempt to solve problems they had unsuccessfully tried to solve using number properties.

During the two groups' units, students chose a number of different footholds, including using all four imaging phases and materials, when unable to solve a problem using number properties. However, during the initial interviews and final assessments, materials were not available, and so students used imaging. The fractions problem, $4 \div 5$, previously discussed in section 4.1.2, illustrates examples of this. Jane used a mental picture image of interlinking cubes, representing chocolate bars, to systematically share the four chocolate bars between five people. Using the mental picture image she had created, she was able to solve the problem. Simon and Ruth used drawn picture images to solve the same problem, having unsuccessfully attempted to use mental picture images. By drawing these images, I would suggest that Simon and Ruth were able to link their images of how they would share the chocolate bars with the formal representation of the problem. Pirie and Kieren (1992) suggest that a reason why students are unable to solve problems using number properties is because their level of understanding is inadequate. The link Simon and Ruth created between their images and the formal representation of the problem allowed them to extend their inadequate level of understanding and correctly solve the problem (Pirie & Kieren, 1992; Pirie & Martin, 2000).

However, three students were unable to solve the problem, despite attempting to use both mental and drawn picture images. Young-Loveridge et al. (2007) report a similar finding in their study of Year 7-8 students' strategies for solving addition problems involving unlike fractions. Their students' recordings of intuitive understandings could not be connected to the formal representation of the problem (Young-Loveridge et al., 2007). Pirie (2002) also discusses the difficulties students face connecting their images of a problem to its formal representation. Students are asked to make a "disconnected mental leap" from the image they have created to using abstract numbers without understanding how the features of their image relate to the numbers (Pirie, 2002, p. 930). This is what the students in this study were probably trying to do, and is a possible reason why they were unsuccessful in solving the problem.

The examples discussed here highlight two issues. The first is the importance of students being able to choose from a full range of mathematical resources to support their strategies for solving problems. The second is the ways teachers support students to make connections between the features of their image and the formal representation of the problem.

6.7 The role of imaging in linking Using Materials and Using Number Properties

The assessment activities suggest that at least some of the students, particularly in group 1, had reached that phase of Using Number Properties, where they no longer needed "an image or a concrete meaning for their mathematical activity" (Pirie & Kieren, 1992, p. 249). Two factors, in particular, seem to have helped these students to reach the top of their climbing walls. First, students' ability to both move between the mathematical resources and follow their own pathways, at their own speeds, seems to have been important in developing their knowledge of the strategy. Second, the students who used number properties to successfully solve all or most of the problems in the final assessment, had moved from creating either mental or drawn picture images, directly related to the context and materials used to explore the problems, to using transformed images, mostly empty number lines. It does not appear to be important whether these images were drawn or mental transformed images. For example, Ruth used imaging extensively during the unit, initially using drawn and mental picture images, and often reaching for the foothold of materials, and later using transformed drawn images. She solved all the final assessment problems correctly, creating a transformed drawn image to solve one, recording her steps as a reminder of what she had done as she solved a second, and reasoning with the numbers and their properties to solve the final two problems. Simon also used a variety of images during the unit (including both mental and drawn transformed images), and materials if he needed to. He described the problems in the final assessment as easy, commenting that he did not need to use an empty number line, but could do them in his head, although he did record numbers as a reminder of what he had done. He was able to choose the most effective strategy to solve each problem, including one strategy, equal additions, that had not been taught during the unit.

Peter only solved two of the four problems correctly. He did not use imaging during the final assessment, even when he found problems difficult. Instead, he relied on rounding to tidy numbers (which sometimes made the problem much more complicated), total place value partitioning, and treating decimal numbers as whole numbers to find answers for three out of the four questions. He commented that he sometimes "got like one off [the correct answer]." I am not suggesting that Peter was not able to solve the problems because he did not use imaging. However, I think it is worth noting that he did not seem to have alternative ways of either solving problems when he got stuck or checking his answers, other than using different numbers.

This chapter has discussed findings from Chapters 4 and 5. The nature of the images used by students has been described, together with the ways these images were used during teacher-led and independent activities. The role of imaging in the process of mathematical thinking has also been discussed, including the use of transformed images and students' preferences for drawn images. The conclusions that I have drawn from these findings, together with the implications of the study, will be discussed in the final chapter.

Chapter 7: Conclusions

This chapter discusses this study's conclusions relating to the nature of imaging and learning in mathematics. A model illustrating the development of students' mathematical understanding is introduced to show the complexity of the process of mathematical thinking. The chapter concludes with a discussion of the implications of this study.

7.1 The nature of imaging

This study has identified four phases of imaging, discussed how these different phases are used, and has highlighted the complexity of the imaging process. Each of the four phases, mental and drawn picture images, and mental and drawn transformed images, contributed to the development of students' understanding, and supported their progress towards the top of their climbing wall for any given problem. While there are similarities between these phases of imaging, there are, however, also important differences.

Students used both picture and transformed images (and both mental and drawn images) to scaffold their learning when progress was blocked by incomplete understanding. Both kinds of images enabled students to approach problems with a 'can-do' attitude, to see themselves as capable learners, and to experience success as mathematicians (Ministry of Education, 2007a). Picture images supported students' developing knowledge about both the meaning of the numbers they were working with (especially decimal numbers) and the strategies they were learning, and were used most frequently in the early stages of the teaching units. Picture images, used in this way, appear to have been essential elements in supporting students to develop their knowledge about the strategies being taught. Picture images were also used when students' understandings were challenged, although the results indicate that this use of picture images generally occurred in the earlier parts of the teaching units (particularly for group 1 students), and when group 1 students attempted to answer the fractions questions in the initial interviews. I would suggest two reasons for this finding. First, students' use of picture images may have been a necessary phase through which most students progressed as they acquired an understanding of the mathematical concepts, and developed confidence in manipulating decimal fractions. And, in the case of the fractions problems, there may not be a suitable alternative image.

The above explanation of picture images is similar to the P-K Image Making stage (Pirie & Kieren, 1989; Pirie & Martin, 2000). Students working at the P-K Image Making stage use action-tied images to solve problems, and their activities are singular and directed. Similarly, in this study the picture images described by students appear to be "essential for thought," in that the problem could not be solved without using an image (Gray & Pitta, 1999b, p. 14). However, the P-K Image Making stage includes materials as well as images. Students are described as working at tasks, physical or mental, that foster initial concepts about the topic. An example is given of students folding sheets of paper to represent fractions (Pirie & Kieren, 1994b; Pirie & Martin, 2000). Picture images, as described in this study, appear to be similar to those described in the STM Using Imaging phase (Hughes, 2002; Ministry of Education, 2008b). However, the STM definition of picture images is narrower than the one I have used in this study because it does not include the use of drawn picture images. The Using Imaging phase refers only to the "visual imagery of absent objects," although this was later extended to include materials that were visible but which could not be manipulated (Hughes, 2002, p. 353).

The use of transformed images as scaffolds differs from the use of picture images in two main respects. First, whereas picture images appear to have been 'essential for thought,' transformed images supported the "creative, critical and metacognitive processes" used by students to make sense of ideas and to actively create knowledge (Gray & Pitta, 1999b; Ministry of Education, 2007a, p. 12). Transformed images enabled students to build on existing knowledge, to explore mathematical solutions, and, through 'progressive mathematization,' develop more formal and efficient strategies (Beishuizen, 2010; Klein et al., 1998). Second, it appears that students may have used transformed images to overcome the "disconnected mental leap" they are often asked to make when moving from using an image of some kind to using number properties (Pirie, 2002, p. 930). Students who used transformed images had developed an understanding of the nature of the numbers they were manipulating and the strategies they were using. They were using this understanding to manipulate numbers, although they were not yet at the stage of using number properties.

Picture and transformed images are both part of students' repertoire of ways of solving problems, although they are used in different ways and for different purposes.

7.2 The nature of learning in mathematics

7.2.1 The nature of learning in mathematics is a complex and fluid process

The findings from this study show that learning in mathematics is a complex and fluid process, with students moving between a number of different mathematical resources to support their developing knowledge. In Figure 7.1, the model illustrates the development of students' mathematical understanding. In using the model, it is important to note that it is based on my observations of what the students in this study said and did. The students themselves were not consciously deciding to use a particular mathematical resource; rather they responded to the complexity of the problem and the level of their understanding. The six mathematical resources discussed in this study are represented in this model. I have deliberately linked the mental and drawn components of the picture and transformed imaging phases to represent the relationship between them. However, as the model illustrates, they can also act as separate phases of imaging. The student at the centre of the model, having analysed the nature of a problem, chooses the mathematical resource they decide is most appropriate to begin to solve the problem. Their subsequent pathway is fluid. Using the metaphor of a climbing wall, students move upwards and, if necessary sideways or downwards, as they reach for suitable hand- and footholds to enable them to find a solution, choosing whichever mathematical resource is the most appropriate to solve a particular aspect of the problem. The six mathematical resources are enclosed within a circle, to encapsulate the dynamic process of students' developing mathematical understanding. This model also signifies that understanding is developing no matter what mathematical resource students are using, or the nature of their individual pathway, as they work towards solving problems using number properties. The model was developed while working with two groups outside their regular classroom. I worked with one group at a time, without the interruptions or teaching rotations of a busy classroom. Nevertheless, I believe the model is useful for teachers working within the complexities of busy mathematics classrooms.

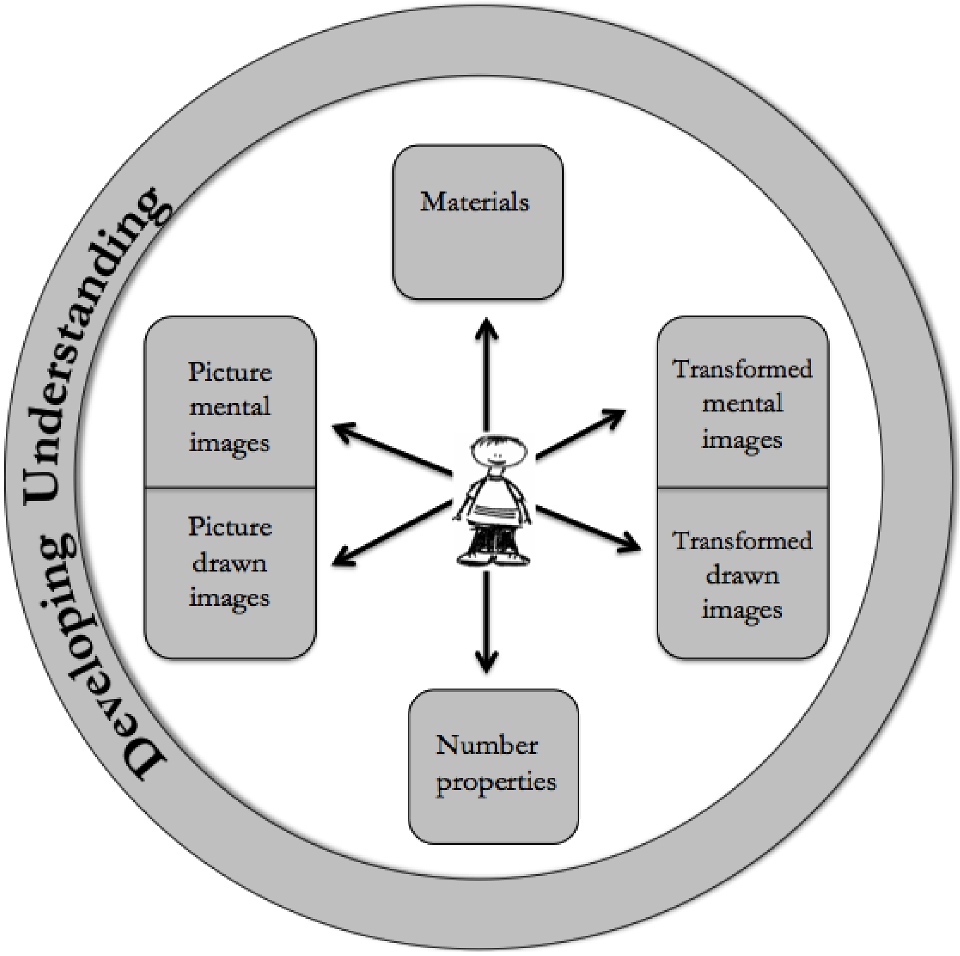


Figure 7.1: A model for the development of students' mathematical understanding

This model differs considerably to the one presented in the STM, which suggests a linear pathway with students moving backwards and forwards ('folding back') between adjacent phases, until, eventually, they are able to solve problems Using Number Properties. Hughes revised the STM in 2008 to recognize that students do move at different rates through the model (Hughes, 2008, private communication from numeracy advisers; see also Appendix F). His revised model is, however, still essentially a linear model. Although the Using Materials and Using Imaging phases can be omitted if students demonstrate understanding of the strategy by Using Number Properties, the movement forwards and backwards through the phases of the revised STM is linear.

7.2.2 Students constructing knowledge

Students in this study employed fluid pathways as they worked towards solving problems using number properties. At the same time, students were constructing their own knowledge, building on prior learning, and, in some instances, developing their own solutions (Confrey, 1990; Ministry of Education, 2007a; von Glasersfeld, 1990, 1993). There appear to have been a number of elements present in this study that facilitated this autonomy. Materials were available for students to use at all times during both teaching units. While students were encouraged to move towards the top of their climbing wall (number properties), they were able to use materials at any time. Students had sufficient time to explore strategies, and concepts. This was particularly important in developing group 1 students' knowledge of decimal place value. As Ann said, she had used decimal fractions before, but had not understood what each number represented and so had often muddled up the numbers and got the answers wrong. As the teacher, I worked to guide students towards using number properties, building on their understanding, listening to their ideas, and questioning their decisions. I found it proved very difficult for students to move to engage with the next phase until they were comfortable to do so, and was counter-productive to try because they almost always reached for the foothold of the phase that represented their current level of understanding. Students in both groups worked collaboratively to solve many problems. They discussed ideas, questioned each other, at times corrected each other, each person contributing to the shared knowledge and learning from it.

7.2.3 The importance of drawn images and written recording

Students in this study expressed a clear preference for using drawn rather than mental images. In articulating this, they were referring mainly, but not exclusively, to transformed drawn images of empty number lines. Students gave a number of reasons for their preference including:

- they could 'see' what the problem was asking them to do;
- they could remember what they had already done;
- they could check that their solution was correct;
- they did not forget what they had done because they had to remember all the steps in their heads (the drawn image was a record of their thinking);
- and they were able to concentrate on the computation necessary to complete the next step.

Students used drawn images to support their strategies for solving problems in a number of other ways. Drawn images allowed students to solve problems they would not otherwise have been able to, as illustrated by group 1's reasons for using transformed drawn images to create problems in the *Create a question* activity. Further, there is evidence of 'progressive mathematization' as a result of students' use of transformed drawn images (Beishuizen, 2010; Klein et al., 1998). Two examples illustrate this. Ruth and Jill were able to use an 'invented,' and more sophisticated, strategy to find the difference between 1.68 and 3.54 because they recorded their thinking on a drawn empty number line. Second, Tom described how he was able to use basic facts to solve a problem, rather than jumping up an empty number line in groups of tens and ones, which is how he had solve previous problems using mental images of empty number lines. This preference for drawn images does not mean that students did not use mental images, or that mental images were not used in the same way as transformed images. One student in particular said she preferred using transformed mental images, and Simon also said that if it was easy he did it in his head (giving his example of 1 + 1).

This preference for drawn images is important in light of the emphasis placed on mental calculation by the NDP (Ministry of Education, 2008b). Mental calculation is emphasized in both the organization of the programme (particularly the focus on small group teaching and oral discussion), and in the teaching and assessment resources. Students select appropriately from a range of mental calculation strategies, and are encouraged to share their thinking orally in a small instructional group.

I claim that the drawn images described in this study can be described as written recording. Yet, the role of written recording within the NDP appears contradictory. *Book 1: The Number Framework* emphasizes the importance of students building meaning by recording their mathematical ideas. Written recording is described as a "thinking tool, a communication tool, and a reflective tool" (Ministry of Education, 2007b, p. 14). In contrast, *Book 3: Getting started* outlines the role of written recording as helping students store information as they solve problems (Ministry of Education, 2008b). The positioning of written recording as one of the five knowledge domains in the NDP gives an implicit message that written recording is simply a tool. However, if students' reasoning is limited to mental reasoning, and oral expression of that reasoning (especially during assessment activities), are we meeting students' needs? The preferences expressed by students in this study would suggest that we are not.

Furthermore, *Book 3: Getting started* states that the "NDP heavily emphasizes flexibility and facility with mental calculation" (Ministry of Education, 2008b, p. 3). Results from this study show students using "flexibility and facility" in solving problems using transformed drawn images, particularly Ruth and Jill. Further, I would suggest that, had the students only been able to use mental calculation, it is likely they would have used a less sophisticated strategy, for example jumping along a number line in groups of ten rather than using derived facts as both Emily and Tom did during the addition and subtraction unit.

7.3 Implications of the study

The findings of this study, particularly the use of four phases of imaging, the fluidity of the pathways followed by students, and their preference for drawn rather than mental images, have implications for my classroom practice, for my teaching colleagues, and for the way the Strategy Teaching Model is used in New Zealand primary classrooms.

One implication is the complexity of how students approach and solve mathematical problems. Many teachers, including teachers in my school, follow the teaching resource material and the phases of the Strategy Teaching Model with its linear backward and forward movement. *Book 3: Getting started* states that the progression to Using Number Properties is "promoted by increasing the complexity of size of the numbers involved, thus making reliance on the material representation difficult and inefficient" (Ministry of Education, 2008b, p. 5). If we follow this method, how do we ensure that students have made the necessary connections between the image and number properties that will enable them to confidently manipulate numbers? I thought I had in 2010, when I taught the *Birthday cakes* activity (Ministry of Education, 2008c). The evidence from the initial interviews suggested otherwise, with only one student in group 1 using number properties to solve the problem. I taught the same strategy again earlier this year, and tried to make certain that the students in my 2012 group were able to describe and explain how the image and the numbers were linked. It seems that encouraging students to use an *image of their choosing*, to record their thinking as they choose, and to move backwards and forwards on a fluid pathway is one way of ensuring that students make the connections between materials and imaging, and imaging and number properties. For teachers, this means supporting students working at different phases for the same activity, and ensuring that materials are always available. It also means giving students time and encouragement to construct their own knowledge, to achieve mathematical success and develop that 'can-do' attitude. This is not easy in a busy mathematics classroom where a teacher is often managing three groups, and working instructionally with two on any given day.

The emphasis on mental calculation and oral reasoning in the Numeracy Development Projects needs to be critically evaluated. The findings from this small-scale study clearly show students' preference for using drawn images, particularly transformed drawn images. In their use of transformed drawn images, it is evident that students in this study were not only calculating in their head, but by using their 'heads,' and by then recording their ideas, they were displaying flexible thought processes (Klein et al., 1998). This has implications for the way we both teach and assess numeracy in New Zealand classrooms. Are we disadvantaging students by not encouraging them to record their thinking? Simon used number properties to solve all four problems in the final assessment, having followed a fluid pathway through the six phases of my model. However, as he developed his solutions, he recorded numbers as signposts to help him remember what he had done. What about students who have difficulty expressing their ideas orally, as a number of students in group 2 did during the stimulated-recall and final assessment interviews? It may be that by recording their ideas, they are able to more clearly demonstrate their understanding. Finally, there are students, like Tom, who by using a recorded image was able to use a more sophisticated strategy than he would have done had he only been allowed to use mental strategies. In my classroom I encourage students to record their ideas, to draw images (either transformed or picture), and to use them as part of the process to solve problems.

This study has focused on some of the complexities of students' thinking as they solve mathematical problems within the confines of investigating the process of imaging within a numeracy classroom. A more interesting question for future investigation might be "What thinking strategies are used by students as they solve mathematical problems?" Such an investigation could focus on the different strands of the mathematics curriculum, as well as a wider range of students, including students in their early years of primary schooling (Years 1-3). In investigating the thinking strategies students use, the study could look into the role of materials, including the effectiveness of the specialized materials often used in NZ mathematics classrooms in comparison to everyday materials (Walls, 2004). It could also explore the role of teachers and peers in developing thinking strategies across the wider mathematics curriculum, and whether the teaching model used in the NDP is used in other areas of mathematics. Finally, it could investigate the nature of the thinking strategies students use when asked to solve problems presented within contexts or settings outside the confines of the Numeracy Development Projects (Bloomfield, 2003). An investigation of this nature would provide insight into the nature of students' thinking that could inform the mathematical teaching practices in primary school classrooms.

In conclusion, this study has investigated and shed light on the nature and complexity of students' thinking as they solve mathematical problems. Of particularly significance for me as a classroom teacher have been the findings related to students' preferences for drawn rather than mental images, the four phases of imaging, and the importance of ensuring students have the time, support and necessary resources to follow individual pathways and actively create knowledge. My challenge is to develop and teach programmes that ensure all students have the time to develop their pathways to mathematical understanding, to connect their actions on materials and images with number properties, and to construct their own knowledge. In this way I will help students become critical thinkers, to develop a 'can-do' attitude and experience success as mathematicians.

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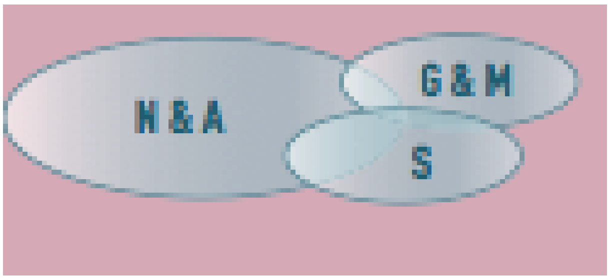
Appendix A: Curriculum Venn diagrams

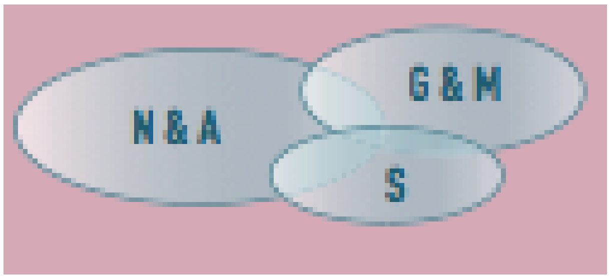
The Venn diagrams illustrate the amount of time that should be spent teaching each mathematics strand. As the diagrams illustrate, the time spent teaching number and algebra varies between 80 per cent of mathematics teaching at Level 1 and 40-60 per cent at Level 4.

Level 1

Level 2



Level 3



Level 4

Appendix B: The Number Framework

(Ministry of Education, 2007b, pp. 15-17)

The Number Framework - Strategies

		Operational Domains		
	Global Stage	**Addition and Subtraction**	**Multiplication and Division**	**Proportions and Ratios**
Counting	**Zero: Emergent**	**Emergent** The student is unable to count a given set or form a set of up to ten objects.		
Counting	**One: One-to-one Counting**	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple addition and subtraction problems.	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple multiplication and division problems.	**Unequal Sharing** The student is unable to divide a region or set into two or four equal parts.
Counting	**Two: Counting from One on Materials**	**Counting - from One** The student solves simple addition and subtraction problems by counting all the objects, e.g., 5 + 4 as 1, 2, 3, 4, 5, 6, 7, 8, 9. The student needs supporting **materials**, like fingers.	**Counting - from One** The student solves multiplication and division problems by counting one to one with the aid of **materials**.	**Equal Sharing** The student is able to divide a region or set into given equal parts using **materials**. With sets this is done by equal sharing of materials. With shapes symmetry (halving) is used.
Counting	**Three: Counting from One by Imaging**	**Counting - from One** The student **images** all of the objects and counts them. The student does not see ten as a unit of any kind and solves multi-digit addition and subtraction problems by counting all the objects.	**Counting - from One** The student **images** the objects to solve simple multiplication and division problems, by counting all the objects, e.g., 4 × 2 as 1, **2**, 3, **4**, 5, **6**, 7, **8**. For problems involving larger numbers the student will still rely on materials.	**Equal Sharing** The student is able to share a region or set into given equal parts by using **materials** or by **imaging** the materials for simple problems, e.g., $\frac{1}{2}$ of 8. With sets this is done by equal sharing of materials or by imaging. With shapes symmetry is used to create halves, quarters, eighths, etc.
Counting	**Four: Advanced Counting**	**Counting On** The student uses counting on or counting back to solve simple addition or subtraction tasks, e.g., 8 + 5 by 8, 9, 10, 11, 12, 13 or 52 – 4 as 52, 51, 50, 49, 48. Initially, the student needs supporting **materials** but later **images** the objects and counts them. The student sees 10 as a completed count composed of 10 ones. The student solves addition and subtraction tasks by incrementing in ones (38, 39, 40, ...), tens counts (13, 23, 33, ...), and/or a combination of tens and ones counts (27, 37, 47, 48, 49, 50, 51).	**Skip-counting** On multiplication tasks, the student uses skip-counting (often in conjunction with one-to-one counting), e.g., 4 × 5 as 5, 10, 15, 20. The student may track the counts using materials (eg. fingers) or by imaging.	

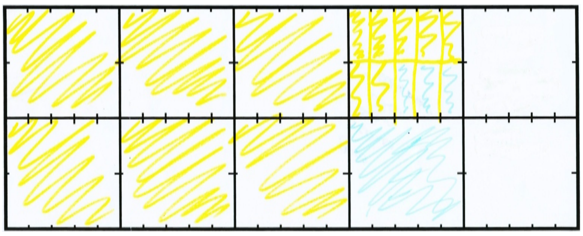
		Operational Domains		
	Global Stage	**Addition and Subtraction**	**Multiplication and Division**	**Proportions and Ratios**
Part-Whole	**Seven: Advanced Multiplicative (Early Proportional) Part-Whole**	**Addition and Subtraction of Decimals and Integers** The student can choose appropriately from a broad range of mental strategies to estimate answers and solve addition and subtraction problems involving decimals, integers, and related fractions. The student can also use multiplication and division to solve addition and subtraction problems with whole numbers. e.g., 3.2 + 1.95 = 3.2 + 2 – 0.05 = 5.2 – 0.05 = 5.15 (compensation); e.g., 6.03 – 5.8 = $\square$ as 6.03 – 5 – 0.8 = 1.03 – 0.8 = 0.23 (standard place value partitioning) or as 5.8 + $\square$ = 6.03 (reversibility) e.g., $\square$ + 3.98 = 7.04 as 3.98 + $\square$ = 7.04, $\square$ = 0.02 + 3.04 = 3.06 (commutativity) e.g., $\frac{3}{4} + \frac{5}{8} = (\frac{3}{4} + \frac{2}{8}) + \frac{3}{8} = 1\frac{3}{8}$ (partitioning fractions) e.g., 81 – 36 = (9 × 9) – (4 × 9) = 5 × 9 (using factors) e.g., 28 + 33 + 27 + 30 + 32 = 5 × 30 (averaging) e.g., $^{+}7 - {}^{-}3 = {}^{+}7 + {}^{+}3 = {}^{+}10$ (equivalent operations on integers)	**Advanced Multiplication and Division** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. These strategies involve partitioning one or more of the factors in multiplication, and applying reversibility to solve division problems, particularly those involving missing factors and remainders. The partitioning may be additive or multiplicative. e.g., 24 × 6 = (20 × 6) + (4 × 6) (place value partitioning) or 25 × 6 – 6 (rounding and compensating) e.g., 81 ÷ 9 = 9, so 81 ÷ 3 = 3 × 9 (proportional adjustment) e.g., 4 × 25 = 100, so 92 ÷ 4 = 25 – 2 = 23 (reversibility and rounding with compensation) e.g., 90 ÷ 5 = 18 so 87 ÷ 5 = 17 r 2 (rounding and divisibility) e.g., 201 ÷ 3 as 100 ÷ 3 = 33 r 1, 200 ÷ 3 = 66 r 2, 201 ÷ 3 = 67 (divisibility rules)	**Fractions, Ratios, and Proportions by Multiplication** The student uses a range of multiplication and division strategies to estimate answers and solve problems with fractions, proportions, and ratios. These strategies involve linking division to fractional answers, e.g., $11 \div 3 = \frac{11}{3} = 3\frac{2}{3}$ e.g., $13 \div 5 = (10 \div 5) + (3 \div 5) = 2\frac{3}{5}$ The student can also find simple equivalent fractions and rename common fractions as decimals and percentages. e.g., $\frac{5}{6}$ of 24 as $\frac{1}{6}$ of 24 = 4, 5 × 4 = 20 or 24 – 4 = 20 e.g., 3:5 as $\square$: 40, 8 × 5 = 40, 8 × 3 = 24 so $\square$ = 24. e.g., $\frac{3}{4} = \frac{75}{100}$ = 75% = 0.75
Part-Whole	**Eight: Advanced Proportional Part-Whole**	**Addition and Subtraction of Fractions** The student uses a range of mental partitioning strategies to estimate answers and solve problems that involve adding and subtracting fractions, including decimals. The student is able to combine ratios and proportions with different amounts. The strategies include using partitions of fractions and "ones", and finding equivalent fractions. e.g., $2\frac{3}{4} - 1\frac{2}{3} = 1 + (\frac{3}{4} - \frac{2}{3}) = 1 + (\frac{9}{12} - \frac{8}{12}) = 1\frac{1}{12}$ (equivalent fractions) e.g., 20 counters in ratio of 2:3 combined with 60 counters in ratio 8:7 gives a combined ratio of 1:1.	**Multiplication and Division of Decimals/ Multiplication of Fractions** The student chooses appropriately from a range of mental strategies to estimate answers and solve problems that involve the multiplication of fractions and decimals. The student can also use mental strategies to solve simple division problems with decimals. These strategies involve the partitioning of fractions and relating the parts to one, converting decimals to fractions and vice versa, and recognising the effect of number size on the answer, e.g., $3.6 \times 0.75 = \frac{3}{4} \times 3.6 = 2.7$ (conversion and commutativity); e.g., $\frac{2}{3} \times \frac{3}{4} = \square$ as $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ so $\frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$ so $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$ e.g., 7.2 ÷ 0.4 as 7.2 ÷ 0.8 = 9 so 7.2 ÷ 0.4 = 18 (doubling and halving with place value).	**Fractions, Ratios, and Proportions by Re-unitising** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve problems involving fractions, proportions, and ratios. These strategies involve using common factors, re-unitising of fractions, decimals and percentages, and finding relationships between and within ratios and simple rates. e.g., 6:9 as $\square$:24, $6 \times 1\frac{1}{2} = 9$, $\square \times 1\frac{1}{2} = 24$, $\square$ = 16 (between unit multiplying); or $9 \times 2\frac{2}{3} = 24$, $6 \times 2\frac{2}{3} = 16$ (within unit multiplying) e.g., 65% of 24: 50% of 24 is 12, 10% of 24 is 2.4 so 5% is 1.2, 12 + 2.4 + 1.2 = 15.6 (partitioning percentages).

		Operational Domains		
	Global Stage	Addition and Subtraction	Multiplication and Division	Proportions and Ratios
Part-Whole	**Seven: Advanced Multiplicative (Early Proportional) Part-Whole**	**Addition and Subtraction of Decimals and Integers** The student can choose appropriately from a broad range of mental strategies to estimate answers and solve addition and subtraction problems involving decimals, integers, and related fractions. The student can also use multiplication and division to solve addition and subtraction problems with whole numbers. e.g., $3.2 + 1.95 = 3.2 + 2 - 0.05$ $= 5.2 - 0.05$ $= 5.15$ (compensation); e.g., $6.03 - 5.8 = \square$ as $6.03 - 5 - 0.8 = 1.03 - 0.8$ $= 0.23$ (standard place value partitioning) or as $5.8 + \square = 6.03$ (reversibility) e.g., $\square + 3.98 = 7.04$ as $3.98 + \square = 7.04$, $\square = 0.02 + 3.04 = 3.06$ (commutativity) e.g., $\frac{3}{4} + \frac{5}{8} = (\frac{3}{4} + \frac{2}{8}) + \frac{3}{8} = 1\frac{3}{8}$ (partitioning fractions) e.g., $81 - 36 = (9 \times 9) - (4 \times 9) = 5 \times 9$ (using factors) e.g., $28 + 33 + 27 + 30 + 32 = 5 \times 30$ (averaging) e.g., $^{+}7 - ^{-}3 = ^{+}7 + ^{+}3 = ^{+}10$ (equivalent operations on integers)	**Advanced Multiplication and Division** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. These strategies involve partitioning one or more of the factors in multiplication, and applying reversibility to solve division problems, particularly those involving missing factors and remainders. The partitioning may be additive or multiplicative. e.g., $24 \times 6 = (20 \times 6) + (4 \times 6)$ (place value partitioning) or $25 \times 6 - 6$ (rounding and compensating) e.g., $81 \div 9 = 9$, so $81 \div 3 = 3 \times 9$ (proportional adjustment) e.g., $4 \times 25 = 100$, so $92 \div 4 = 25 - 2 = 23$ (reversibility and rounding with compensation) e.g., $90 \div 5 = 18$ so $87 \div 5 = 17$ r 2 (rounding and divisibility) e.g., $201 \div 3$ as $100 \div 3 = 33$ r 1, $200 \div 3 = 66$ r 2, $201 \div 3 = 67$ (divisibility rules)	**Fractions, Ratios, and Proportions by Multiplication** The student uses a range of multiplication and division strategies to estimate answers and solve problems with fractions, proportions, and ratios. These strategies involve linking division to fractional answers, e.g., $11 \div 3 = \frac{11}{3} = 3\frac{2}{3}$ e.g., $13 \div 5 = (10 \div 5) + (3 \div 5) = 2\frac{3}{5}$ The student can also find simple equivalent fractions and rename common fractions as decimals and percentages. e.g., $\frac{5}{6}$ of 24 as $\frac{1}{6}$ of $24 = 4$, $5 \times 4 = 20$ or $24 - 4 = 20$ e.g., 3:5 as $\square$: 40, $8 \times 5 = 40$, $8 \times 3 = 24$ so $\square = 24$. e.g., $\frac{3}{4} = \frac{75}{100} = 75\% = 0.75$
	Eight: Advanced Proportional Part-Whole	**Addition and Subtraction of Fractions** The student uses a range of mental partitioning strategies to estimate answers and solve problems that involve adding and subtracting fractions, including decimals. The student is able to combine ratios and proportions with different amounts. The strategies include using partitions of fractions and "ones", and finding equivalent fractions. e.g., $2\frac{3}{4} - 1\frac{2}{3} = 1 + (\frac{3}{4} - \frac{2}{3}) = 1 + (\frac{9}{12} - \frac{8}{12}) = 1\frac{1}{12}$ (equivalent fractions) e.g., 20 counters in ratio of 2:3 combined with 60 counters in ratio 8:7 gives a combined ratio of 1:1.	**Multiplication and Division of Decimals/ Multiplication of Fractions** The student chooses appropriately from a range of mental strategies to estimate answers and solve problems that involve the multiplication of fractions and decimals. The student can also use mental strategies to solve simple division problems with decimals. These strategies involve the partitioning of fractions and relating the parts to one, converting decimals to fractions and vice versa, and recognising the effect of number size on the answer, e.g., $3.6 \times 0.75 = \frac{3}{4} \times 3.6 = 2.7$ (conversion and commutativity); e.g., $\frac{2}{3} \times \frac{3}{4} = \square$ as $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ so $\frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$ so $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$ e.g., $7.2 \div 0.4$ as $7.2 \div 0.8 = 9$ so $7.2 \div 0.4 = 18$ (doubling and halving with place value).	**Fractions, Ratios, and Proportions by Re-unitising** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve problems involving fractions, proportions, and ratios. These strategies involve using common factors, re-unitising of fractions, decimals and percentages, and finding relationships between and within ratios and simple rates. e.g., 6:9 as $\square$:24, $6 \times 1\frac{1}{2} = 9$, $\square \times 1\frac{1}{2} = 24$, $\square = 16$ (between unit multiplying); or $9 \times 2\frac{2}{3} = 24$, $6 \times 2\frac{2}{3} = 16$ (within unit multiplying) e.g., 65% of 24: 50% of 24 is 12, 10% of 24 is 2.4 so 5% is 1.2, $12 + 2.4 + 1.2 = 15.6$ (partitioning percentages).

Appendix C: Materials used during the study

Decipipes showing 0.23

Linked cubes showing fourteen-tenths



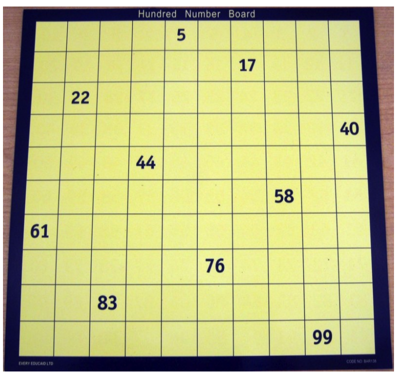
Decimats showing tenths and hundredths

Counters showing 4 tens and 5 ones

Animal strips showing 4 tens and 5 ones

Counting beans – the canisters showing the tens

Bundled ice cream sticks showing 4 tens and 5 ones



A hundreds board

Appendix D: Lesson planning

Group 1			
Day/ Date	**Main content**	**Key activities**	**Key resources**
Day 1/ 24 May	Developing an understanding of decimal place value – tenths and hundredths: Key ideas and knowledge check.	What is a decimal fraction? Identify symbols for any fraction. Counting forwards and backwards in 10ths. Diagnostic snapshot – how many 10ths in a	NDP Book 7: p. 35-38 Decipipes Linked cubes
Day 2/ 25 May	Knowledge: reading decimal numbers, meaning of the decimal point, significance of zero as a place holder, counting forwards and backwards in 10ths and 100ths. There are different names for decimal fractions. Decimal place value extends to the left and right by multiplying or dividing by 10. Modelling decimal numbers, and explaining the relationship between the number and model.	Place value hangman. Making and describing decimal fractions as mixed numbers, improper fractions, remainders, decimal numbers. Encourage students to image improper fractions. *Pipe music with decimals* – using decipipes to explain 10ths and 100ths. *Pipe music with decimals*: what does one-quarter look like when made on decipipes; building decimal numbers on decipipes and describing the number; imaging to predict what number would look like.	NDP Book 7: pp. 38-40 (*Pipe music with decimals*) Number flip chart Linked cubes Decipipes
Day 3/ 26 May	Knowledge: reading decimal numbers, meaning of the decimal point, significance of zero as a place holder, counting forwards and backwards in 10ths and 100ths. Introduce 1000ths. Decimal numbers bring out different considerations of the decimal system. Using place value to add 10ths.	Place value hangman *Pipe music with decimals*: Which number is larger? Is 0.37 or 0.4 larger? Reinforcing significance of zero as a place holder. *Pipe music with decimals*: introduce numbers where 1000ths are needed. *Pipe music with decimals*: pose addition problems that can be modelled on decipipes.	Number flip chart NDP: Book 7: p. 40 Decipipes
Day 4/ 27 May	Knowledge: as Day 3 Using place value to add decimal numbers using 10ths and 100ths. Open-ended problem solving using 10ths and 100ths (and extension 1000ths)	Place value hangman Using imaging/materials to solve problems like 0.5 + 0.37 Posing problems where answers are wrong because of a misunderstanding of place value: imaging to explain why 0.5 + 0.8 does not equal 0.13, and why 0.5 + 0.68 does not equal 0.68 (materials available if necessary). Open-ended problem solving: Make 0.5 or 2.07. Make 2.07 – extension, use 10ths, 100ths, 1000ths Make 0.5 use 10ths and 100ths, use materials if necessary, recording working.	NDP: Book 7: p. 40 *New Zealand Curriculum maths: Book 7* Decipipes

Day/ Date	Main content	Key activities	Key resources
Day 5/ 31 May	Knowledge: reading decimal numbers, counting forwards and backwards, including using 1000ths, ordering decimal numbers and positioning numbers on a number line. Subtraction/difference problems using 10ths and 100ths.	Place value hangman – numbers to include 1000ths. Position 10 numbers between 4.2 and 4.3. Posing subtraction problems that have the wrong answer to highlight common misunderstandings – using a calculator to subtract money. *Pipe music with decimals*: Pose subtraction problems (e.g. difference between 0.56 and 0.3), materials available if necessary, observing and recording strategies students choose to solve problems. Key problems: difference between 2.5 and 0.89.	Number flip chart Pre-printed empty number lines. *New Zealand Curriculum maths: Book 7* NDP: Book 7, pp. 40-41. Decimats
Day 6/ 1 June	Knowledge: Ordering decimal numbers (including 1000ths) by positioning on a number line. Knowledge: everyday problems involving decimal numbers. Solving difference problems using 10ths and 100ths by reversing and adding.	Position 10 numbers between, for example, 4.21 and 4.23. Everyday problems involving decimals that highlight common misunderstandings: which is bigger a 1.5 litre bottle or a 335 ml bottle? Solving difference problems by reversing and adding (for example, the difference between 0.8 and 0.67). Using materials, and introducing decimats. Students working with a partner. Encouraging imaging – made decipipes or students looking at but not colouring in decimats. Key problems: difference between 1.4 and 0.6, 0.8 and 0.67, 0.66 and 0.9, 1.6 and 0.81, 0.7 and 0.54.	Pre-printed empty number lines. *New Zealand Curriculum maths: Book 7* NDP: Book 7: pp. 40-41 Decipipes Decimats
Day 7/ 2 June	Knowledge: how many 10ths or 100ths are in a number. Using imaging to add and subtract decimal numbers. Solving problems using decimal numbers.	Knowledge: work in pairs, each with different number (for example, 5.67), and explain how many 10ths and then 100ths in the whole of the number. Teacher-led imaging: problems modelled on decipipes which students are able to see but not touch. Work with a partner to explain how they would solve the problem. Students solving problems mentally. Key problems: 0.75 + 0.6, difference between 1.53 and 0.8, difference between 0.413 and 0.89. Problem solving using Arb word problem.	NDP: Book 7: pp. 40-41 Decipipes Decimats Assessment Resource Banks: Maths: NM1300 (www.nzcer.org.nz).

Day/ Date	Main content	Key activities	Key resources
Day 8/ 3 June	Developing knowledge of the relationship between decimal numbers. Additional support to two members of the group, adding and subtracting decimal numbers.	Make a millionth – starting with a very large piece of paper, create, first 10ths, then 100ths, 1000ths etc., by folding and cutting. Explain how each decimal fraction is made and its relationship with the whole. *Pipe music with decimals*: reversing subtraction problems and adding. Using materials, imaging as appropriate and also focusing on interpreting story problems. Key problems: difference between 0.67 and 0.8, 0.54 and 0.7.	4 A3 or A2 sheets of newsprint glued together. NDP: Book 7: pp. 40-41 Decipipes Whiteboards
Day 9/ 7 June	Knowledge: adding decimal numbers; rounding decimal numbers. Revision: using place value, and using reversing to solve subtraction problems. Rounding and compensating to solve addition and subtraction problems.	One decimal place loopy. Rounding decimal numbers to the nearest whole number. Using number properties (or imaging) to reverse subtraction problems or use place value. Key problems: difference between 3.8 and 4.69. *Pipe music with decimals*: teacher-led (initially using materials and moving to imaging and number properties as appropriate) to round and compensate to solve addition problems. Key problems: 1.09 + 0.68, 1.6 + 0.93.	Loopy cards – Secondary Numeracy Project (nzmaths.co.nz).
Day 10/ 8 June	Knowledge: rounding decimal numbers to the nearest whole number, tenth or hundredth; positioning decimal numbers on a number line. Solving problems: use taught strategies to solve problems, choosing the most appropriate strategy for each problem.	Knowledge: using whiteboards and number lines to round decimal numbers and position on a number line. Describing and justifying decisions to group. Solving problems: solve three story problems by identifying key words and then choosing the most appropriate strategy from the ones taught. Work independently and then share strategies with partner. If able to solve mentally using number properties, record just the answer. Otherwise use imaging or, if necessary materials to solve problems. If students use number properties and record just the answer, they must explain their reasoning to their partner, who must record their thinking.	Whiteboards Solving problems sheet Decipipes or decimats

Day/ Date	**Main content**	**Key activities**	**Key resources**
Day 11/ 9 June	Knowledge: ordering decimal numbers; rounding to 2 decimal places. Problem solving	Ordering decimal numbers using the times of 1500 metre Olympic runners (Arb NM0116). Rounding decimals to 2 decimal places – working individually to check that all are rounding accurately and able to justify their decisions. Problem solving: *Create a question*: students to work in pair to create three questions (one for each of the strategies taught) to match given answer. Once questions created using numbers, students to attempt to write a story problem to match the equation.	Book 8: Teaching number sense and algebraic thinking: Rounding decimals, p. 21. nzmaths.co.nz – Material master 8-14. nzcer.org.nz: Arb NM0116 nzmaths.co.nz: Problem solving, Level 4: *Create a question* (http://www.nzmaths.co.nz/resource/create-question?parent_node=)
Day 12/ 10 June	Knowledge: revision of knowledge taught during the unit. Problem solving	Knowledge quiz: work with partner, points for correct answer and for explaining and justifying answer. Questions on all aspects of knowledge taught during the unit. Problem solving: complete questions from day 11. Swap problems with another group. They solve one of the questions and identify the strategy used to write the problem.	Book 8: Teaching number sense and algebraic thinking, Confusing fractions and decimals, p. 20. nzmaths.co.nz: Materials masters 8-12. nzmaths.co.nz: Problem solving, Level 4: *Create a question*.

Group 2			
Day/ Date	**Main content**	**Key activities**	**Key resources**
Day 1/ 21 June	Revision and assessment that students were confident solving problems using Stage 5 outcomes already taught. Specifically, solving addition problems by partitioning numbers to make groups of 10, interpreting word problems and writing equations, using groupings of 10 and doubles to solve addition problems. Throughout the unit: encouraging students to express ideas orally using mathematical language.	Knowledge: Add 7 cards: use numbers that add to 10 and doubles to solve problems. Strategy: *Up over 10*, Book 5, p. 28, Using Number Properties *Adding in parts*, Book 5, p. 28, Using Number Properties. *On and off the train*, *Figure it out*, Level 2, number 2, p. 14, Using Imaging (tens frames, and ice cream sticks) Using the problem progression in the AC-EA addition and subtraction unit plan.	Playing cards NDP: Book 5 *Figure it out*, Level 2, number 2 nzmaths.co.nz, numeracy unit plans, AC-EA, addition and subtraction Ice cream sticks (in bundles of 10)
Day 2/ 22 June	Developing confidence counting backwards and forwards in groups of 10 from any number to 100. Partitioning to solve addition problems. Adding groups of ten, keeping first number together rather than partitioning both numbers. Adding tens and ones, by keeping the first number together and adding first the ones and then the tens on to it.	Knowledge: Add 7 cards – finding groupings of 10 or doubles. Counting forwards and backwards in groups of 10 from any number to 100 by imaging a hundreds board. Strategy: *Adding 10s*, Book 5, p. 23 – using imaging. *Adding ones and tens*, Book 5, p. 24, using materials, and using imaging. Problems used three-digit numbers to encourage students to use number properties.	Playing cards NDP: Book 4 NDP: Book 5 Hundreds boards Ice cream sticks
Day 3/ 23 June	Applying basic fact knowledge of addition and subtraction. Counting forwards and backwards in groups of 10. Adding 10s and 1s.	Knowledge: Basic facts – *Bowl a fact*, Book 4, p. 35. *Leap frog*, *Figure it out*, Level 2, number 2, p. 12. Progression: Using Materials (counting on 100s board), Using Imaging by first looking at numbers but not touching, and then imaging the reverse (blank side). *Adding tens and ones*, Book 5, imaging shielded materials.	NDP: Book 4 Whiteboards and pens *Figure it out*, Level 2, number 2 NDP: Book 5 Hundreds boards Ice cream sticks
Day 4/ 24 June	Knowledge of place value of numbers to 1 million, and reading numbers to 1 million. Confidently counting forwards in 10s from any number to 100. Adding tens and ones using change unknown problems.	Short session today because of other school activities. Knowledge: *Number hangman*, Book 4 *Missing ones and tens*, Book 5, p. 25. Using hundreds boards (materials and imaging) to solve change unknown problems, starting with the 3 tens + ? = 8 tens, and moving to 36 + ? = 66.	NDP: Book 4 NDP: Book 5 Number flip chart Hundreds boards

Day/ Date	Main content	Key activities	Key resources
Day 5/ 28 June	Reading numbers to 1 million, particularly the pattern of reading numbers; saying the number 1, 10, 100 more of less than a number. Going over a hundred while counting forwards in groups of 10. Adding tens and ones, with an emphasis on keeping the first number together (and not partitioning both numbers)	Place value hangman with numbers to 1 million. Reading numbers to 1 million, with a focus on counting forwards and backwards in 10s. Counting forwards in 10s – counting over a hundred. *Adding 10s and 1s*, Book 5, p. 24, using materials and imaging. Focus on keeping the first number together (a number of the students in the group are partitioning both numbers and adding the 10s before adding the 1s). Students choose from a variety of materials to model problems. Move to imaging made materials.	Number flip chart (to count forwards and backwards) Place value houses Numeracy money NDP: Book 5. Hundreds boards Ice cream sticks Counters Film canisters of beans Animal strips
Day 6/ 29 June	Reading numbers to 1 million; saying the number 1, 10, 100 more of less than a number. Developing confidence counting backwards and forwards from any number to 100 (to develop fluency, and to encourage children to keep numbers together, rather than partitioning), including change unknown problems involving counting in tens. Solving problems using materials, imaging and number properties by adding 10s and 1s.	Place value hangman with numbers to 1 million. Reading numbers to 1 million, with a focus on counting forwards and backwards in 10s. *Adding 10s and 1s*, Book 5, p. 24, using a variety of materials (students choose materials they prefer). Focus on students keeping one of the numbers together, and adding the ones and the tens on to it (rather than partitioning both numbers). Move to imaging by shielding. Students solving problems mentally, and explaining strategies orally. Include 3-digit numbers to encourage use of number properties, and extend students.	Revolving hundreds board Number flip chart (to count forwards and backwards) NDP: Book 5 Hundreds boards Ice cream sticks Counters Film canisters of beans Animal strips
Day 7/ 30 June	Continuing to develop fluency counting forwards and backwards in 10 and 1s, focusing on imaging. Counting in 10s from any number to 1000 (especially where rollover to next hundred). Solving problems by imaging or using number properties by adding 10s and 1s. Exploring and experimenting with numbers, to find patterns.	Counting forwards and backwards in 10s by imaging a hundreds board. Counting forwards in 10 from any number to 1000, predicting and then checking on calculator *Adding 10s and 1s*, Book 5, p. 24, using shielded materials (imaging) and number properties. Reversing problems so that the biggest addend is first. Problem solving – *Reversing numbers* – take a 2-digit number and reverse it. Add the numbers together. What do you notice about the numbers? Materials available if necessary, or using imaging or number properties.	Hundreds boards Calculators NDP: Book 5 www.nzmaths.co.nz/resource/reversing-numbers?parent_node= Hundreds boards Ice cream sticks Counters Film canisters of beans Animal strips

Day/ Date	Main content	Key activities	Key resources
Day 8/ 1 July	Developing fluency counting forward and backwards in 10s. Adding hundreds, tens and ones that do not involve renaming. Introduce addition problems involving change unknown.	Counting forwards and backwards, imaging a hundreds board and describing what is happening to the numbers on the hundreds board. *Adding 10s and 1s* – number properties – adding two 3-digit numbers, and extending to add 4-digit numbers. Making connections between how single digit numbers are added with adding hundreds and thousands. *Missing ones and tens*, Book 5, p. 25 – introduce with materials (students in group have a choice of materials) – first making both numbers with materials and comparing the difference, and then making one number and predicting how much will be added on (before modelling with materials).	Hundreds boards NDP: Book 5 Counters Ice cream sticks Animal strips Film canisters of counting beans
Day 9/ 5 July	Developing fluency counting backwards in 10s. Solve change unknown problems using materials and moving to imaging.	Counting backwards using a hundreds board to image if necessary. Adding multiples of 10 to 100, and making connections adding multiples of 100 to 1000. *Missing ones and tens*, Book 5, p. 25 – model again using choice of materials, or imaging/number properties if confident. *Trimming trees*, solving change unknown problems using more abstract materials (hundreds board).	NDP: Book 5 *Figure it Out*, Number Sense and Algebraic Thinking, Level 2-3, book 2, p.6, *Trimming trees* Counters Ice cream sticks Animal strips Hundreds boards
Day 10/ 6 July	Counting backwards in 10s (especially over a hundred), in preparation for the introduction of subtraction strategies. Stage 5 knowledge – multiples of 100 that add to 1000. Solve change unknown problems, using imaging or number properties. Introduce subtraction of 10s and 1s that does not involve renaming.	Counting backwards in 10s – predicting and then checking with a calculator. *Missing ones and tens*, Book 5, p. 25 – imaging made ice cream sticks to support student absent the day before. *Trimming trees* – change unknown problems involving numbers from 101-200, using imaging/number properties, or materials if necessary Introduce *Subtracting ones and tens*, Book 5, p. 24 – using materials, and focusing on interpreting subtraction story problems.	Calculators NDP: Book 5 *Figure it Out*, Number Sense and Algebraic Thinking, Level 2-3, book 2, p.6, *Trimming trees* Hundreds boards Counters Ice cream sticks Animal strips Film canisters

Day/ Date	**Main content**	**Key activities**	**Key resources**
Day 11/ 7 July	Making connections between multiples of 10 that add to 100 and multiples of 100 that add to 1000. Know groupings within 100 (Stage 5 knowledge). Subtracting 10s and 1s, focus on making sure students only partition one number.	Quick fire – numbers that add to 100 or 1000. Students posing own problems involving subtraction of 2-digit numbers. Other members of the group solve them. Progression from students imaging by looking at made materials, to using number properties. Materials available to support students who need them.	Counters Ice cream sticks Animal strips Film canisters
Day 12/ 8 July	Knowledge to support subtraction – counting back in 10s. Problem solving using the three strategies taught.	Counting back in 10s – predicting and then using a calculator. Revision of addition of 2-digit numbers using a story board (students write story problem, equation and model with materials). Independently solving a number of problems that represent the three strategies taught. Use of number properties encouraged, but materials available if necessary.	Story board master Calculators Counters Ice cream sticks Animal strips Film canisters
Day 13/ 12 July	Provide support for students who had struggled with problems the previous day. Provide extension for students who had solved problems confidently.	Students completing from activity from previous session, or additional problems. Working with two students to support strategies using change unknown problems. Working with Emily to encourage her to record thinking. *Weka wobble*, *Figure it out*, Level 2, book 1, p. 11, number 3 – extension activity using problems that involve making groups of 10 to solve addition problems.	Counters Ice cream sticks Animal strips Film canisters Numeracy money *Figure it out*, Level 2, book 1, p. 11 - *Weka wobble*

Appendix E: Interview, assessment and teaching questions

Group 1: Initial interview questions

Each question was put into the context of a story problem. A copy of the equation was visible as students worked to solve the problems.

Question 1 26 + ? = 82

Question 2 24 x 6 = ?

Question 3 4 ÷ 5 = ?

Question 4 You eat three-sevenths of a cake. If there are eighteen candles on the three-sevenths you eat, how many candles are on the whole cake?

Question 4 (alternative) You eat two-fifths of a cake. If there are twelve candles on the two-fifths you eat, how many candles are on the whole cake?

There were two options for question 4 because of the complexity of the tables in the first option. Option 1 was included because I felt there would be some students in the group who would need a problem of this complexity to enable them to show the strategy they were using. Option 2 was included because I knew the complexity of the tables would stop students (who understood the strategy) showing their knowledge.

Group 1: Final assessment questions

Each question was put into the context of a story problem. A copy of the equation was visible as students worked to solve the problems.

Question 1 3.21 + 1.96 =

Question 2 8.65 + 4.2 =

Question 3 6.48 – 3.92 =

Question 4 6.13 – 5.8 =

Group 2: Initial interview questions

Questions (with the exception of question 1) were put into the context of story problems. A copy of the equation was visible as students worked to solve the problems.

Question 1 3 + 8 + 6 + 7 + 2 + 4 =

Question 2 47 + 5 =

Question 3 53 – 5 =

Question 4 43 + 30 =

Group 2: Final assessment questions

Each question was put into the context of a story problem. A copy of the equation was visible as students worked to solve the problems.

Question 1 42 + 37 =

Question 2 24 + ? = 69

Question 3 83 – 61 =

Question 4 523 + 246 =

Question 5 783 – 151 =

Examples of the questions asked during the units

What were you thinking as you solved that problem?

What did you do in your head as you solved that problem?

Did you use a mental image to help as you solved the problem? If so, what was the image you used, and how did the image help?

Why did you choose to use that strategy?

How did you know that strategy would be effective?

Is there a better way to solve that problem? If so, what would that be?

How did you know…?

Why did you choose to manipulate the materials the way you did?

How did you add those numbers together/subtract those numbers?

Why did you decide to solve the problem the way you did?

What could you do first?

What would you do next (as students work to solve a problem)?

How did using materials, imaging made materials, or creating visual images help you to solve the problem?

What would ? look like if you made it with materials?

Appendix F: Hughes revised teaching model